Metric Models for Type Theory

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Type Theory

Type Theory is an alternative foundation for mathematics

- Implementations

- **Used for machine-assisted theorem proving** e.g. the Four Colour Theorem, Fermat's Last Theorem, (ongoing), mathlib…

Type Theory

Agda is based on Martin Löf type theory

Assumptions

Formula

$$\Gamma \vdash t : A$$

Proof

*The **type** A is a **formula** and the **program** t of that type is the constructive **proof** of the corresponding formula.*

Categorical Models of Type Theory

The program t is an arrow between objets Γ and A

Arrow

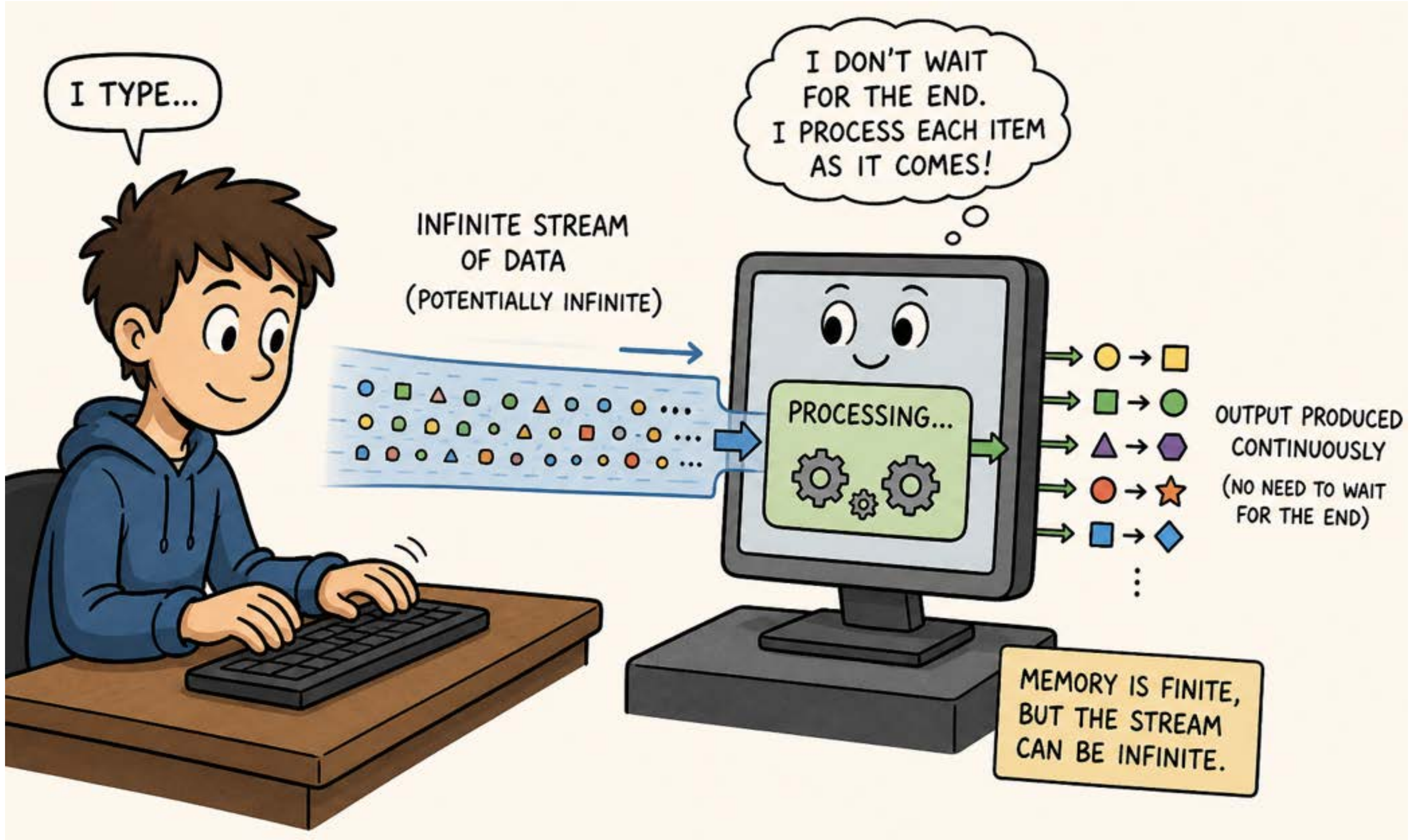
$$\Gamma \xrightarrow{t} A$$

Object

"A type is a formula and the program of that type is the constructive proof of the corresponding formula. "

Streams (Coinductive Types)

A **stream** is an infinite sequence of values
that have to be processed as the data comes in



Streams in Type Theory

Mathematically, a stream is an infinite sequence of values in a type A

$$\mathbb{N} \to A$$

Productive

- $\text{repeat}\ x = x :: \text{repeat}\ x$ ✅
- $\text{bad}\ x = \text{bad}\ x$ ❌
- $\text{nats}\ x = 0 :: \text{map}\,(+1)\ \text{nats}\ x$ ✅

Not Productive

Metric Models of Streams

We equip the set of streams with a bisected ultrametric

$$d(x,y) = \begin{cases} 0 & x = y \\ 2^{-n} & \text{where } n \text{ is the first index where } x_n \neq y_n \end{cases}$$

Properties of Ultrametrics

- All triangles are isosceles with the two longer sides equal
- Balls are nested or disjoint
- Every point inside an open ball is at its center

Categories of Metric Spaces

Objects are bisected ultrametric spaces and arrows are non-expansive functions

Metric Spaces

$$(X, d_X) \xrightarrow{f} (Y, d_Y)$$

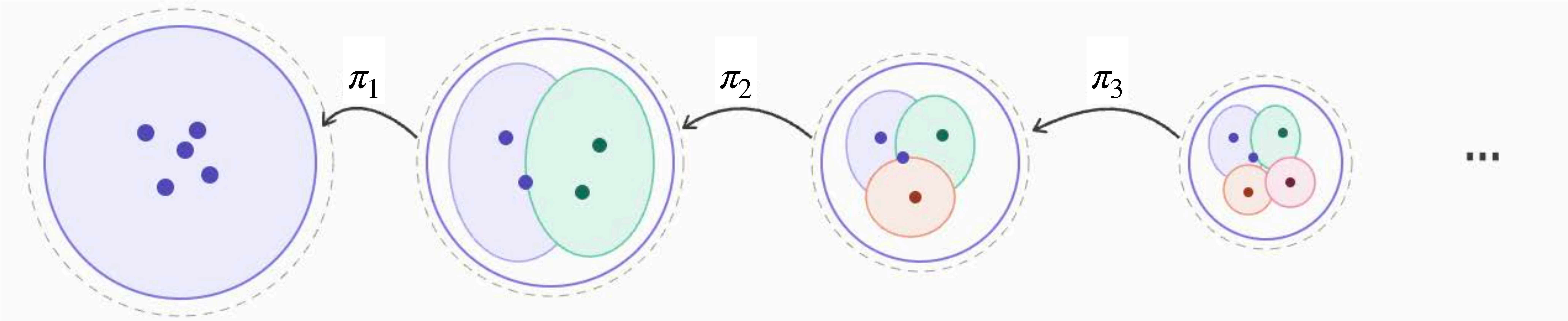
Non-expansive maps

Properties

- Non-expansive programs are **productive**
- Contractive programs are **causal**

Metrics into Presheaves

A bisected ultrametric space (M, d) induces to the sequence $\{M/ =^n\}_{n \in \omega}$



At level n, equivalence classes are 2^{-n}-balls
$[x]_n = \{y \in M \mid d(x, y) \leq 2^n\}$

Restrictions forget last refinement:
merge sub-balls into their parent ball
$\pi_n : M/ =^n \to M/ =^{n-1}$

Metrics and the Categories of Presheaves

There is an adjunction between the category of metric spaces and the category of presheaves over ω

Model of type theory

$$\begin{tikzcd} \boldsymbol{PSh}(\omega) \arrow[r, shift right=2, "G"'] \arrow[r, phantom, "\perp"] & \boldsymbol{BiUlt} \arrow[l, hook', shift right=2, "F"'] \end{tikzcd}$$

F: Every bisected ultrametric space (M, d) induces a sequence
G: The limit of every sequence is a metric space

What about Hilbert spaces?

Definition. The set of streams x satisfying $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ is a Hilbert space called ℓ^2.

Simple Question. Can we do something with it?

Don't have answers, only more problems:

- all streams do not form a Hilbert space.
- the inner product is very likely not an ultrametric.