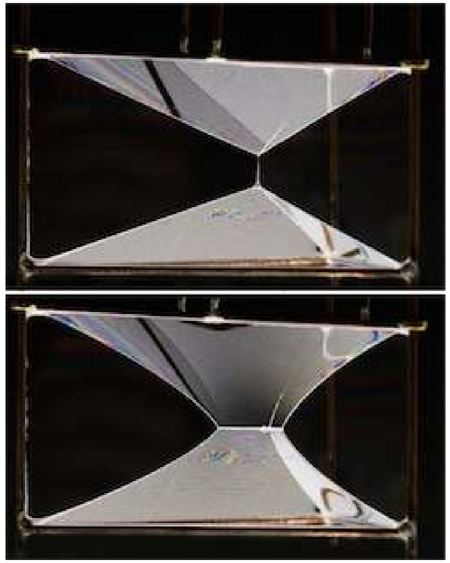
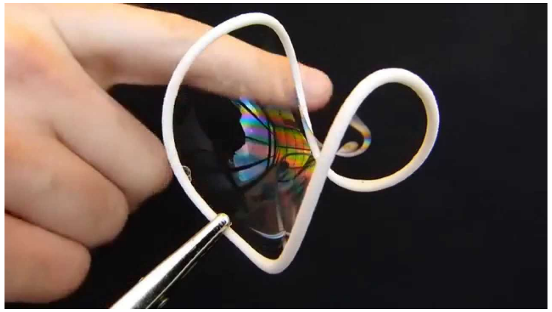
# Quantum minimal surfaces and discrete Painlevé equations





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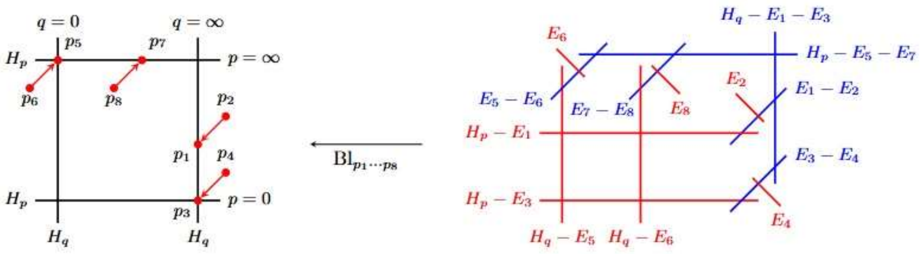
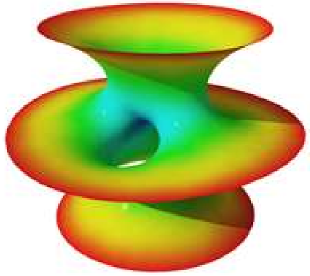
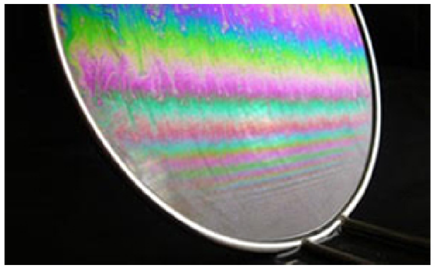
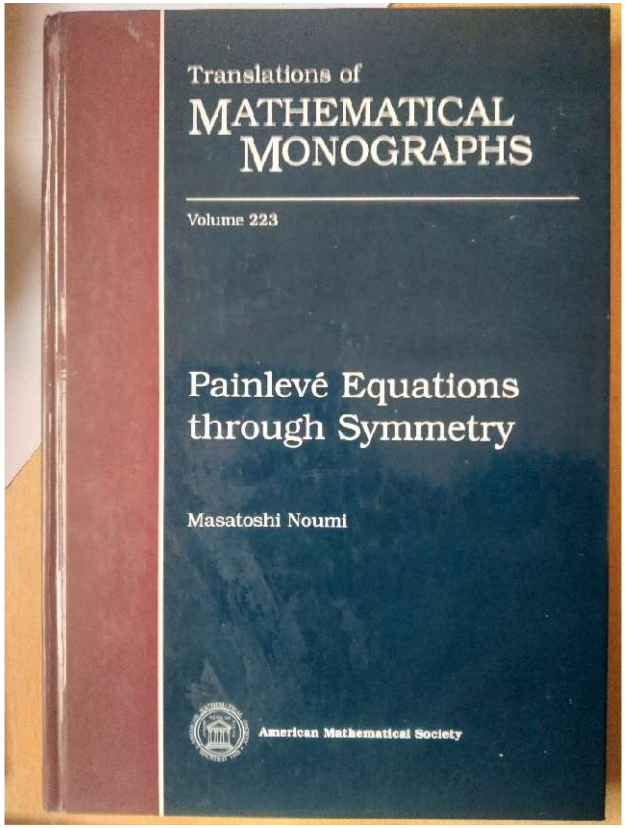




# Dedicated to the memory of Masatoshi Noumi

野海　正俊



# Painlevé equations: nonlinear special functions

The six Painlevé equations $P_I$–$P_{VI}$ are as follows:

$$\frac{d^2 w}{dz^2} = 6w^2 + z, \qquad P_I$$

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha, \qquad P_{II}$$

$$\frac{d^2 w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \qquad P_{III}$$

$$\frac{d^2 w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}, \qquad P_{IV}$$

$$\frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \qquad P_V$$

$$\frac{d^2 w}{dz^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\frac{dw}{dz} + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right), \qquad P_{VI}$$

with $\alpha$, $\beta$, $\gamma$, and $\delta$ arbitrary constants. The solutions of $P_I$–$P_{VI}$ are called the *Painlevé transcendents*. The six equations are sometimes referred to as the Painlevé transcendents.

Problem: Find all $F$ such that the ODE

$$\frac{d^2 w}{dz^2} = F\left(z, w, \frac{dw}{dz}\right)$$

has no movable branch points in all its solutions $w(z)$.

# Example: First Painlevé equation $P_I$

$$\frac{d^2w}{dt^2} = 6w^2 + t \qquad (P_I)$$

Kovalevsky - Painlevé analysis:

Near movable pole $w(t) \sim \dfrac{1}{(t-t_0)^2}$

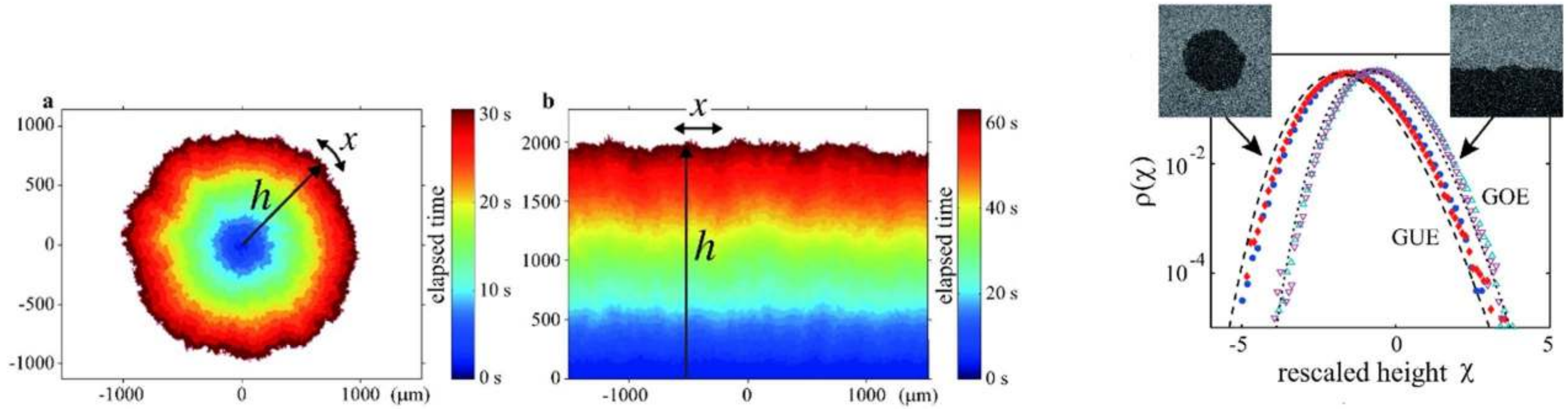
$\longrightarrow$ Laurent series $w = \sum_{n=0}^{\infty} c_n (t-t_0)^{n-2}$, $c_0 = 1$.

All solutions are meromorphic, transcendental functions of $t$.

cf. Weierstrass $\wp$-function: $\wp'' = 6\wp^2 - \frac{1}{2} g_2$.

# Applications of Painlevé equations:

Scaling similarity solutions of PDEs (solitons), random matrices, orthogonal polynomials, 2D quantum gravity, statistical mechanics, probability & statistics,...




Example ①: Correlation functions of Bose gas

$\rightsquigarrow$ solutions of $P_{V}$ (Jimbo-Miwa-Mori-Sato)

Example ②: liquid crystal growth - KPZ equation

$\rightsquigarrow$ Tracy-Widom solution of $P_{II}$ (Takeuchi-Sano - Sasamoto-Spohn)

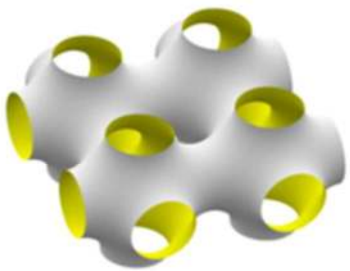
# Quantum minimal surfaces (Arnlind-Hoppe-Kontsevich; cf. Cornalba & Taylor)

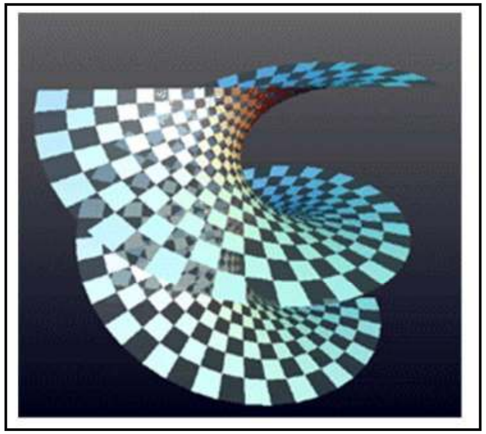
Minimal surfaces can be characterised as maps $\mathbf{x} : \Sigma \to \mathbb{R}^d$ that extremise the Schild functional

$$S[\mathbf{x}] = \int_\Sigma \sum_{j<k} \{x_j, x_k\}^2 \, \omega,$$

where $\Sigma$ is a surface with symplectic form $\omega$ and associated Poisson bracket $\{\bullet, \bullet\}$, and $(x_j)_{j=1,\ldots,d}$ are coordinates on $\mathbb{R}^d$. The Euler-Lagrange equations obtained from the action $S$ are

$$\sum_{j=1}^{d} \{x_j, \{x_j, x_k\}\} = 0, \qquad k = 1, \ldots, d.$$





$$\{q, p\} = 1 \quad (\omega = dp \wedge dq)$$

$$\rightsquigarrow$$

$$[Q, P] = i\hbar \cdot 1$$

**Canonical quantization (above)**

**Quantization of minimal surfaces:**

Operators $X_j$ acting on Hilbert space $\mathcal{H}$, subject to

$$\sum_{j=1}^{d} [X_j, [X_j, X_k]] = 0, \quad k = 1, \ldots, d.$$

# Quantum curves from embeddings in 4D

Special case: Riemann surface $F(z_1, z_2) = 0$ in $\mathbb{C}^2$

$\Downarrow$

$d=4$: Minimal surface $\Sigma$ in $\mathbb{R}^4$ $\begin{pmatrix} z_1 = x_1 + i x_2 \\ z_2 = x_3 + i x_4 \end{pmatrix}$

$F=0 \Rightarrow \{z_1, z_2\} = 0$

constant curvature metric $\Rightarrow \{\bar{z}_1, z_1\} + \{\bar{z}_2, z_2\} = iK$

$\left.\begin{matrix} \{z_1, z_2\} = 0 \\ \{\bar{z}_1, z_1\} + \{\bar{z}_2, z_2\} = iK \end{matrix}\right\}$ 1st order system $\Rightarrow$ 2nd order Euler-Lagrange equations

Quantization: Operators $Z_1, Z_2$ on $\mathcal{H}$ with $F(Z_1, Z_2) = 0$,
so $[Z_1, Z_2] = 0$;

and require $[Z_1^\dagger, Z_1] + [Z_2^\dagger, Z_2] = \varepsilon \cdot \mathbb{1}$, where $\varepsilon \sim 2\hbar \in \mathbb{R}$

MAIN EXAMPLE: "Quantum parabola" $Z_2 = (Z_1)^2$: set $Z_1 = W$, $Z_2 = W^2$.

Then require $[W^\dagger, W] + [(W^\dagger)^2, W^2] = \varepsilon \cdot \mathbb{1}$.

# Discrete $P_I$ equation

Introducing $\mathcal{H} = \{ |n\rangle \mid n \in \mathbb{Z}_{\geq 0} \}$ with $W|n\rangle = w_n |n+1\rangle$, the quantum parabola condition on $W$ implies that $v_n = |w_n|^2$ satisfies

$$v_n - v_{n-1} + v_{n+1} v_n - v_{n-1} v_{n-2} = \varepsilon$$

which is the total difference of the 2nd order equation

$$v_n (v_{n+1} + v_{n-1} + 1) = \varepsilon (n+1) \quad (\text{"Discrete } P_I\text{"}).$$

(Fixing an extra constant from the semiclassical limit $\varepsilon = 2\hbar \to 0$, which gives approximation $v_n \simeq \frac{1}{4}\left(\sqrt{1+8\varepsilon(n+1)} - 1\right)$.)

The 2nd order difference equation is referred to as a discrete $P_I$ ($dP_I$) equation, because a suitable finite difference limit $v_n \simeq v(nh)$ as $n \to \infty$, $h \to 0$ with $t = nh$ fixed gives the continuous Painlevé I ODE.

# Unique positive solution

PROBLEM: Show that $\forall \varepsilon > 0$ the $d P_I$ equation

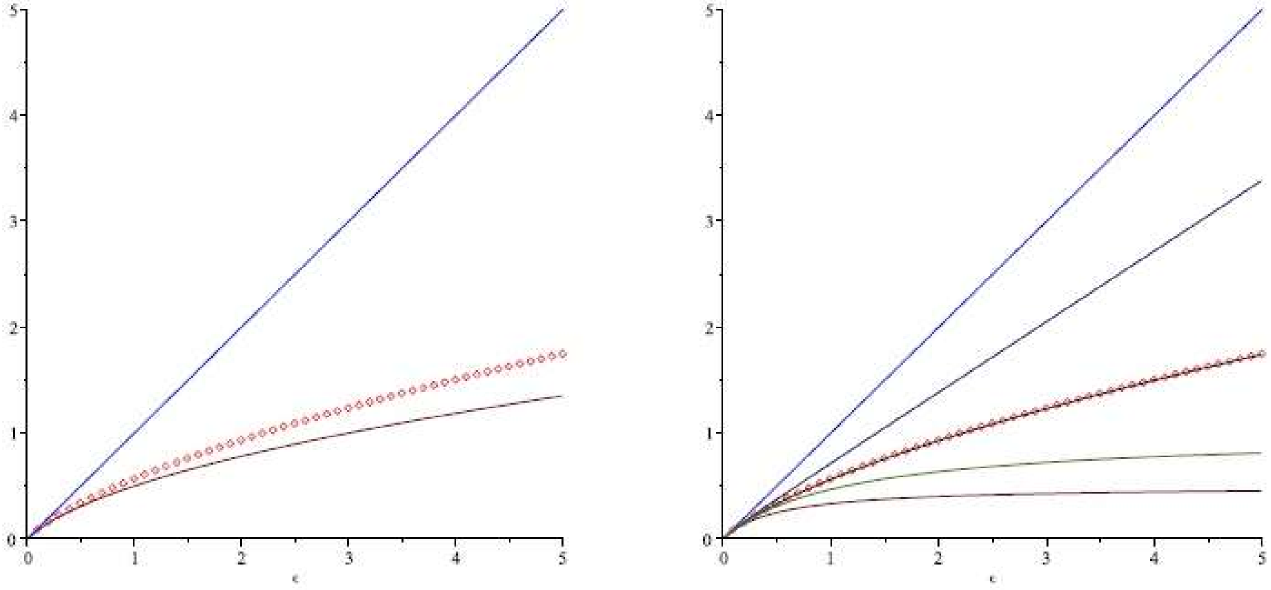
$$v_{n+1} + v_{n-1} + 1 = \frac{\varepsilon(n+1)}{v_n}$$

has a solution with $v_{-1} = 0$, $v_0 > 0$ as initial values, such that $v_n = |w_n|^2 > 0$ $\forall n \geq 0$, and this is unique.

IDEA: Consider sequences $\underline{u} = (u_n)$ with $u_{-1} = 0$ and $u_n \geq 0$ for $n \geq 0$, subject to requirement of finite norm $\|\underline{u}\| := \sup_{n \geq 0} \frac{u_n}{(n+1)\varepsilon} < \infty$, and show mapping $\underline{u} \longmapsto T\underline{u}$ has unique fixed point, where

$$T u_n = \frac{\varepsilon(n+1)}{u_{n+1} + u_{n-1} + \underline{1}}.$$

# Asymptotics of positive solution



Left panel: numerical computation of $V_0(\varepsilon)$ (red dots) compared with $\varepsilon$ (blue line) and approximation $(\sqrt{1+8\varepsilon}-1)/4$.
Right panel: Same, but with smooth interpolation, and pair of upper/lower bounds.

Map $\underline{u} \longmapsto T\underline{u}$ is almost a contraction mapping, giving a sequence of upper/lower rational function approximations to each $V_n(\varepsilon)$.
In particular, for $n=0$ have complete expansion

$$V_0(\varepsilon) \sim \varepsilon - 2\varepsilon^2 + 12\varepsilon^3 - 112\varepsilon^4 + \dots \quad \text{as} \quad \varepsilon \to 0.$$

QUESTION: How to characterize the solution further?

# Complex geometry of Painlevé equations

$$\begin{tikzcd}[column sep=small]
 & & & & & & & A_7^{(1)} \arrow[dr] & \\
A_0^{(1)} \arrow[r] & A_1^{(1)} \arrow[r] & A_2^{(1)} \arrow[r] & A_3^{(1)} \arrow[r] \arrow[dr] & A_4^{(1)} \arrow[r] \arrow[dr] & A_5^{(1)} \arrow[r] \arrow[dr] & A_6^{(1)} \arrow[r] \arrow[dr] \arrow[ur] & A_7^{(1)\prime} \arrow[dr] & A_8^{(1)} \\
 & & & & D_4^{(1)} \arrow[r] & D_5^{(1)} \arrow[r] \arrow[dr] & D_6^{(1)} \arrow[r] \arrow[dr] & D_7^{(1)} \arrow[r] \arrow[d, bend left] & D_8^{(1)} \arrow[d] \arrow[d, bend left] \\
 & & & & & & E_6^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_8^{(1)}
\end{tikzcd}$$

Sequence of blowups of $\mathbb{P}^1 \times \mathbb{P}^1$ $\rightsquigarrow$ Smooth rational surface $\mathcal{X}$

Sakai : Continuous and discrete Painlevé equations
(2nd order non-autonomous ODE/O$\Delta$E)

$\updownarrow$

$(\mathcal{X}, D)$ → $R$ (surface type)

$(\mathcal{X}, D)$ → $R^\perp$ (symmetry type)

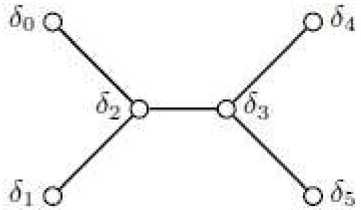
$\mathcal{X}$: Smooth rational surface

$D$: effective divisor

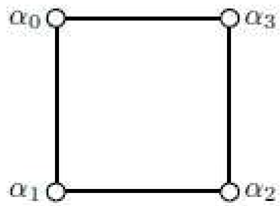
($R$, $R^\perp$ both inside $E_8^{(1)}$.)

# Bäcklund transformations of $P_{V} \longleftrightarrow dP_{I}$ (quantum parabola)

Affine type Dynkin diagrams:




$D_5^{(1)}$




$A_3^{(1)}$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{(w-1)^2}{t^2}\left(\tilde{\alpha} w + \frac{\tilde{\beta}}{w}\right) + \frac{\tilde{\gamma}}{t} w + \tilde{\delta}\frac{w(w+1)}{w-1}, \qquad (P_{V})$$

Applying Sakai's approach to the $dP_{I}$ equation yields $R = D_5^{(1)}$, $R^{\perp} = A_3^{(1)}$: $\mathcal{X}$ corresponds to space of initial conditions for continuous $P_{V}$. In fact, we have $v_n$ associated with a sequence of solutions of $P_{V}$ (as above) with parameters $\tilde{\alpha} = \frac{(n+1)^2}{18}$, $\tilde{\beta} = -\frac{1}{18}$, $\tilde{\gamma} = -\frac{(n+1)}{3}$, $\tilde{\delta} = -1/2$, and

$$v_n(\varepsilon) = \frac{1}{w(t)-1} \quad \text{with} \quad \varepsilon = \frac{1}{3t}.$$

# Classical solutions of $P_{\mathrm{V}}$ and $dP_{\mathrm{I}}$

Although generic solutions are transcendental, on certain hyperplanes in parameter space there can be classical solutions. For $P_{\mathrm{V}}$, these have been found in terms of Whittaker/Kummer functions (Masuda). The above $\tilde{\alpha}, \check{\beta}, \check{\gamma}$ values lie in a suitable union of planes in $\mathbb{R}^3$, and there are classical solutions with $v_{-1} = 0$ and $v_0$ satisfying a Riccati equation:

$$3\varepsilon^2 \frac{dv_0}{d\varepsilon} = \varepsilon(1+2v_0) - v_0 - v_0^2 .$$

Linearize $\rightsquigarrow$ 1-parameter family of classical solutions of $dP_{\mathrm{I}}$.

<u>Theorem</u>: Initial conditions. $v_{-1} = 0$, $v_0 = \frac{1}{2}\left( \frac{K_{5/6}(1/6\varepsilon)}{K_{-1/6}\left(\frac{1}{6\varepsilon}\right)} - 1 \right)$

yield the unique positive solution of the quantum parabola $dP_{\mathrm{I}}$.

# OUTLOOK:

- Positive solutions of dP/qP equations and orthogonal polynomials.
- Other quantum minimal surfaces from rational curves: $Z_2^r = Z_1^s$
  $\gcd(r,s)=1$ e.g. $(1,3)$ or $(2,3)$.
- Analogue of Sakai classification for higher order dP equations?
  cf. Noumi–Yamada systems.