Applications of Adiabatic Flux Insertion to Quantum Many-Body Systems: A Pedagogical Introduction

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This presentation file is based on what was used in the actual talk at #CMPCity2020, but slightly revised and modified (including correction of typos)

Talk 1 (today)

Applications of Adiabatic Flux Insertion to Quantum Many-Body Systems: A Really Pedagogical Introduction

M. O. and T. Senthil, PRL 96, 060601 (2006)

Talk 2 (Friday 26 June)

Adiabatic vs Sudden Flux Insertion and Nonlinear Electric Conduction

Haruki Watanabe and M.O., arXiv:2003.10390
Haruki Watanabe, Yankang Liu, and M. O., arXiv:2004.04561

Vector Potential: U(1) Gauge Field

Global U(1) symmetry in Quantum Mechanics
enhanced to U(1) gauge symmetry

$$\psi(\vec{r}) \to \psi(\vec{r})e^{i\theta} \quad \Longrightarrow \quad \psi(\vec{r}) \to \psi(\vec{r})e^{i\theta(\vec{r})}$$

Replace derivatives by "covariant derivative"

$$\vec{\nabla}\psi(\vec{r}) \quad \Longrightarrow \quad \left(\vec{\nabla} - i\vec{A}(\vec{r})\right)\psi(\vec{r})$$

$$\psi(\vec{r}) \to \psi(\vec{r})e^{i\theta(\vec{r})}$$

$$\vec{A}(\vec{r}) \to \vec{A}(\vec{r}) + \vec{\nabla}\theta(\vec{r})$$

covariant derivative

$$\left(\vec{\nabla} - i\vec{A}(\vec{r})\right)\psi(\vec{r})$$

is gauge invariant

Meaning of Covariant Derivative

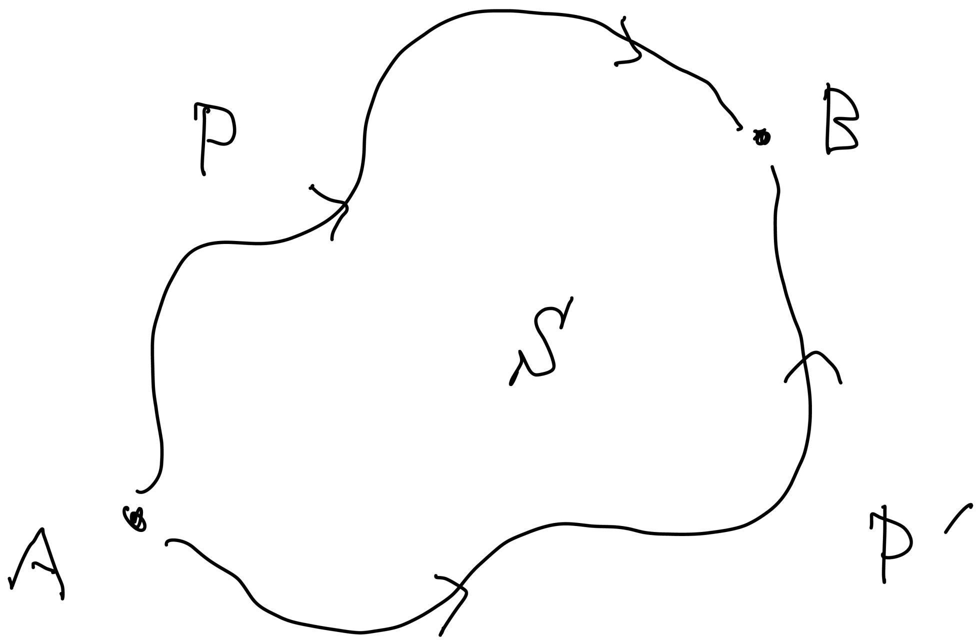
$$\partial_j \psi(\vec{r}) = \lim_{\delta \to 0} \frac{\psi(\vec{r} + \delta \vec{e}_j) - \psi(\vec{r})}{\delta}$$

$$(\partial_j - iA_j)\,\psi(\vec{r}) = \lim_{\delta \to 0} \frac{\psi(\vec{r} + \delta \vec{e}_j) - e^{i\vec{A}(\vec{r})\cdot\delta\vec{e}_j}\psi(\vec{r})}{\delta}$$

"parallel transport"

Even when there were no vector potential initially,
we can introduce a non-zero vector potential by
a gauge transformation = local change of the phase
Before comparing wavefunctions at two points, we need
the corresponding phase change ("parallel transport")

Path Integral



extra phase

$$\exp\left(i\int_P \vec{A}(\vec{r})\cdot d\vec{r}\right)$$

due to the parallel transport along the path

$$\exp\left(i\int_P \vec{A}(\vec{r})\cdot d\vec{r} - \int_{P'} \vec{A}(\vec{r})\cdot d\vec{r}\right) = \exp\left(i\oint_{\partial S} \vec{A}(\vec{r})\cdot d\vec{r}\right)$$

$$\oint_{\partial S} \vec{A}(\vec{r})\cdot d\vec{r} = \int_S \mathrm{rot}\vec{A}\cdot d\vec{n}$$ Stokes' theorem

Gauge Invariance

$\vec{B} = \mathrm{rot}\vec{A}$ ("curvature" = magnetic field) is gauge invariant

$$\mathrm{rot}\vec{A}' = \mathrm{rot}\left(\vec{A}' + \vec{\nabla}\theta\right) = \mathrm{rot}\vec{A}$$

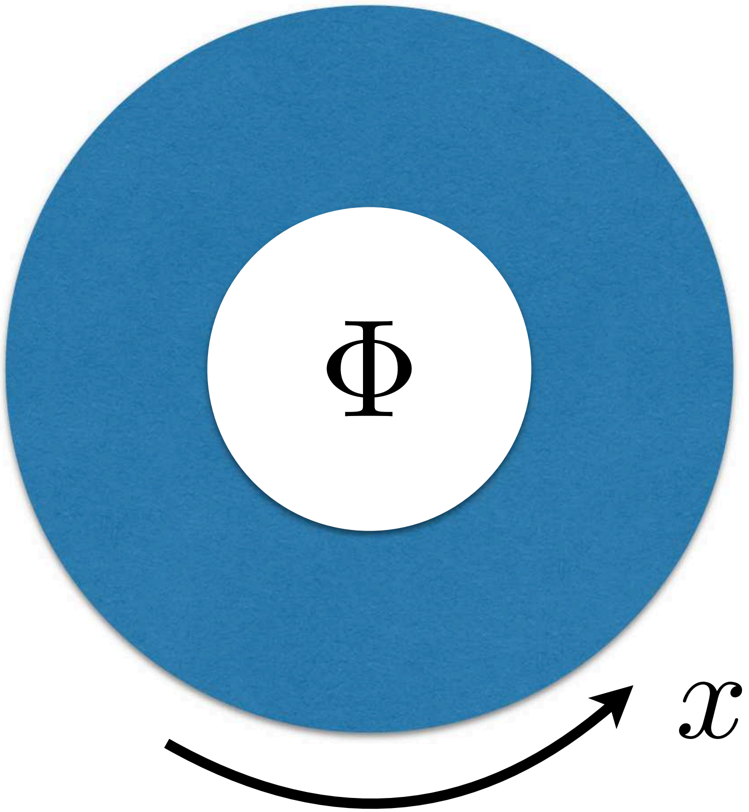
$$\oint_{\partial S} \vec{A}(\vec{r}) \cdot d\vec{r} = \int_S \mathrm{rot}\vec{A} \cdot d\vec{n} \;\; = \int_S \vec{B} \cdot d\vec{n} = \Phi(S)$$

phase difference = magnetic flux through the enclosed area

Only the gauge-invariant magnetic (and electric) field is physical

Vector potential has a gauge ambiguity and must be unphysical (just a mathematical trick) — right?

Aharonov-Bohm Effect



particles do not touch
the magnetic field directly
⇒ no effect within classical mech

But quantum interference is still affected ⇒

Aharonov-Bohm effect

Quantum system defined on the annulus does depend on the flux, except when the Aharonov-Bohm phase is

$$\Phi = 2\pi \times \text{integer}$$

Unit Flux Quantum

I have implicitly chosen the units so that

$$\hbar = 1 \qquad\qquad e = 1$$

Covariant derivative ⇔ kinetic momentum

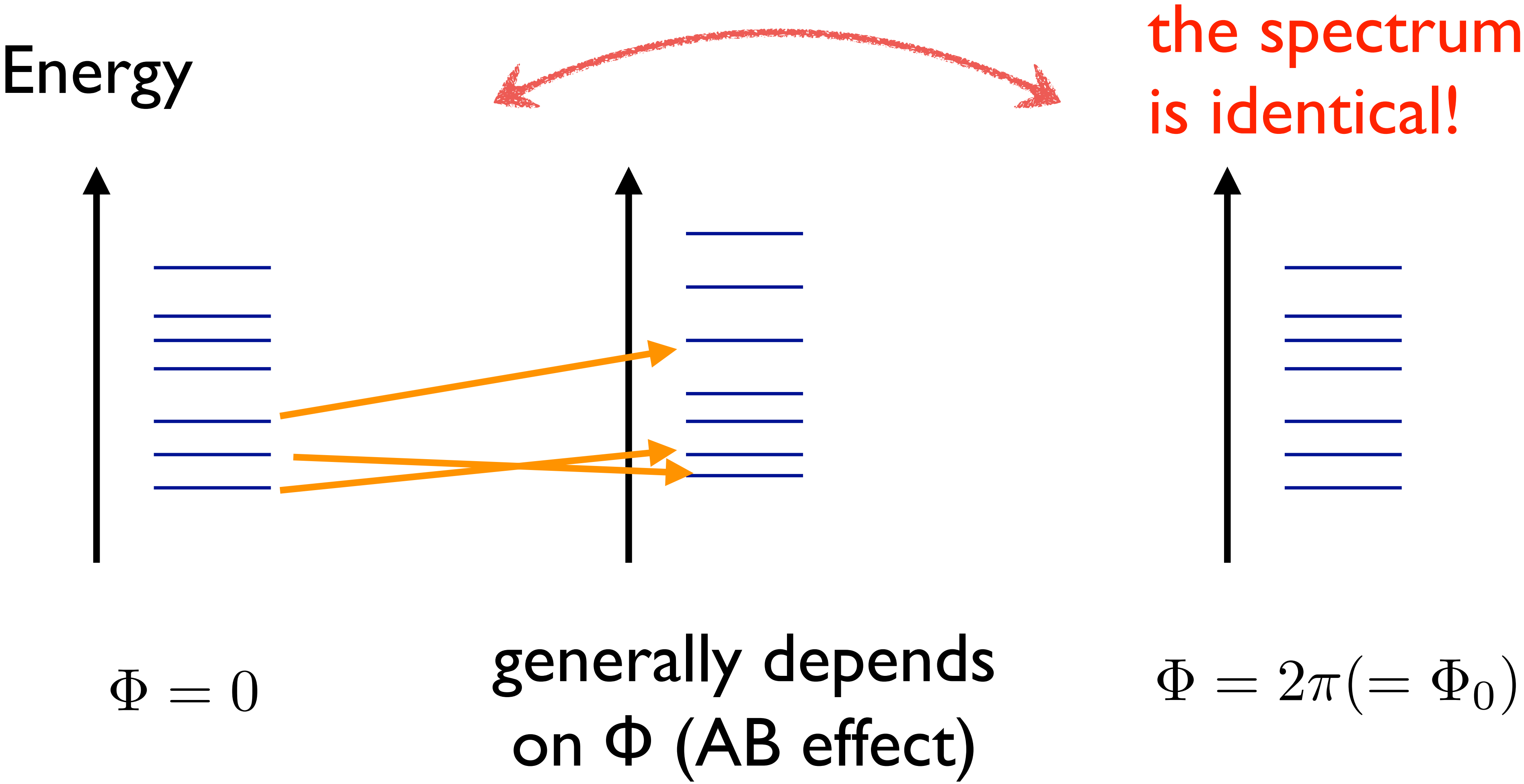
$$\left(-i\hbar\vec{\nabla} - e\vec{A}(\vec{r})\right)\psi(\vec{r})$$

$$\exp\left(i\frac{e}{\hbar}\oint_{\partial S}\vec{A}(\vec{r})\cdot d\vec{r}\right) = \exp\left[2\pi i\frac{\Phi(S)}{\Phi_0}\right]$$

$$\Phi_0 = \frac{h}{e} = 4.136\times 10^{-15}\ \mathrm{Wb}$$

(twice the "unit flux quantum" commonly used in superconductivity literature)

Spectrum of the Hamiltonian



Nevertheless $\mathcal{H}(\Phi = 2\pi) \neq \mathcal{H}(\Phi = 0)$

Large Gauge Transformation

If the Aharonov-Bohm flux is an integral multiple of the unit flux quantum it can be eliminated by a topologically nontrivial ("large") gauge transformation

$$\psi(\vec{r}) \to \psi(\vec{r})e^{i\theta(\vec{r})} \qquad \theta(\vec{r}) = 2\pi\frac{x}{L_x}$$

phase is multivalued but wavefunction is unique

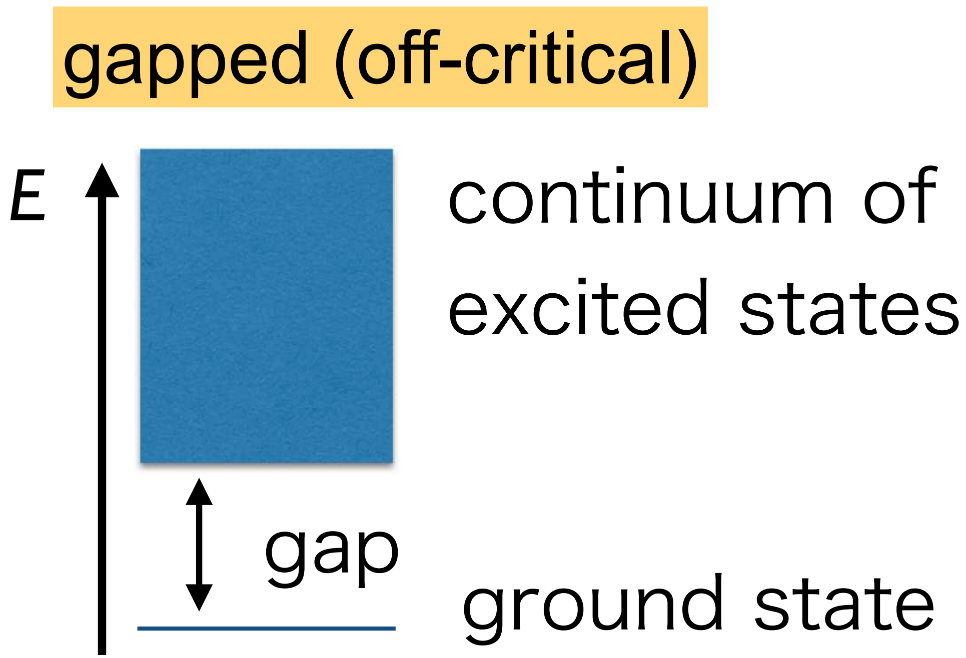
For a many-body Hamiltonian on a lattice

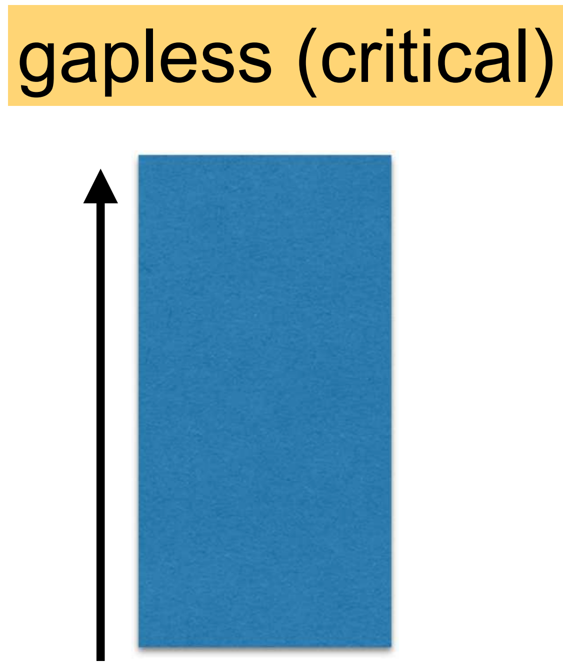
$$\mathcal{H}(\Phi = 2\pi) = U_x{}^{-1}\mathcal{H}(\Phi = 0)U_x$$

$$U_x = \exp\left(\frac{2\pi i}{L_x}\sum_{\vec{r}} x n_{\vec{r}}\right)$$

Quantum Many-Body Systems

Quantum fluctuations can drive the system at T=0 into different quantum phases, and cause quantum phase transitions between quantum phases





Adiabatic Flux Insertion

Let us consider a gapped many-body system, and assume that the gap does not close by the AB flux Φ

nontrivial assumption, but generally true for **insulators** in $d \leqq 2$ (to be discussed later)

physically reasonable even for $d \geqq 3$ and I am not aware of a counterexample within short-range Hamiltonians but can't prove either

Under the assumption, we can insert a unit flux quantum adiabatically: ground state remains ground state

After the Flux Insertion

Because of the adiabaticity, starting from the ground state at $\Phi=0$, we must come back to the ground state at $\Phi=2\pi$

The spectrum of the Hamiltonian must be identical to the initial one by the large gauge invariance

$$\mathcal{H}(\Phi = 2\pi) = {U_x}^{-1}(\Phi = 0)U_x$$

So, we just come back to the same ground state as the initial state?

But sometimes you CANNOT come back to the same state ⇒ ground-state degeneracy!

(or the system is actually gapless)

Fractionalization

Condensed matter: made of protons, neutrons, electrons…
all the **constituent particles** have
(integral multiples of) the **unit charge e**

But some systems have "*fractionalized quasiparticles*"
which carry a **fraction of the unit charge** e
These are collective excitations of many constituent particles (electrons), not "broken pieces" of an electron

We can generalize the notion of "charge" beyond the electric charge, for any ***locally conserved quantity***
e.g. S^z+[electric charge/(2e)] is integer for an electron
so "spinon" or "holon" are fractionalized w.r.t. this "charge"
(introduce a fictitious gauge field coupled to the "charge")

Gapped Fractionalized Phase

We assume that

- microscopically the system is made of the constituent particles with the unit charge
- the system is gapped (and the gap is stable against Φ)
- the system has quasiparticle/quasihole with a fractional charge $\pm\, p/q$
- we can create a quasiparticle/quasihole pair by a local perturbation
- quasiparticle/quasihole can be moved freely

  (i.e. they are not fractons)

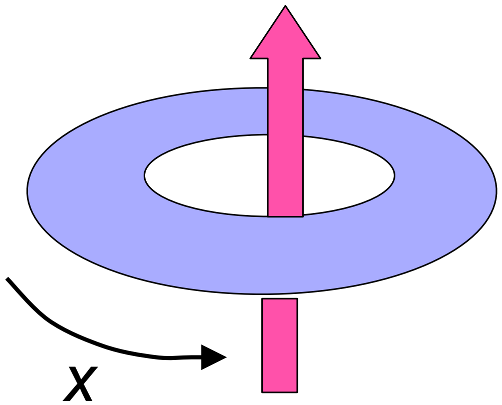
And see what are the consequences, using a gedankenexperiment

M.O. and T. Senthil, PRL 2006

Flux Insertion Operation

Define $\mathcal{F}_x$

adiabatic insertion
of unit flux quantum



uniform gauge:
vector potential

$$A_x = 0 \to \frac{\Phi_0}{L_x}$$

adiabatic time evolution

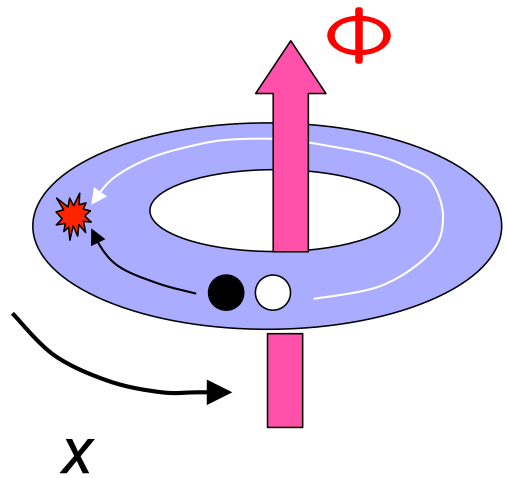
$$\mathcal{F}_x = \mathcal{T} e^{-i \int_0^T \mathcal{H}(A_x = \frac{\Phi_0 t}{T L_x}) dt}$$

$\mathcal{F}_y$ is defined similarly for the *y* direction

Pair Creation/Annihilation

define $\mathcal{T}_x$

as a time evolution w.r.t.
certain time-dep. Hamiltonian
representing



creation of
quasiparticle/hole pair

↓

"dragging" the quasiparticle/hole to
$\pm x$ direction

↓

pair-annihilate the quasiparticle/hole

AB Effect for the Quasiparticle

With a unit flux quantum through the hole, there is no AB effect for the constituent particles (electrons) because the AB phase is 2π × integer, which is unobservable

Nevertheless, for the quasiparticle carrying the fractional charge *p*/*q*, the unit flux quantum is "nontrivial", giving the AB phase = 2π *p*/*q*

$$\mathcal{T}_x(\Phi_0)\mathcal{F}_x = \mathcal{F}_x\mathcal{T}_x(0)e^{2\pi i p/q}$$

quasiparticle dragging in the presence of the unit flux quantum

(relation among 3 operators → can't say anything yet)

AB Effect for Quasiparticle

Because Tx is defined by an appropriate time evolution in terms of the microscopic Hamiltonian,

$$\mathcal{H}_\lambda(\Phi_0) = {U_x}^{-1}\mathcal{H}_\lambda(0)U_x \qquad \mathcal{T}_x(\Phi_0) = {U_x}^{-1}\mathcal{T}_x(0)U_x$$

$$\mathcal{T}_x(\Phi_0)\mathcal{F}_x = \mathcal{F}_x\mathcal{T}_x(0)e^{2\pi i p/q}$$

$$\Rightarrow \quad \mathcal{T}_x(0)\tilde{\mathcal{F}}_x = \tilde{\mathcal{F}}_x\mathcal{T}_x(0)e^{2\pi i p/q}$$

$$\tilde{\mathcal{F}}_x \equiv U_x\mathcal{F}_x$$

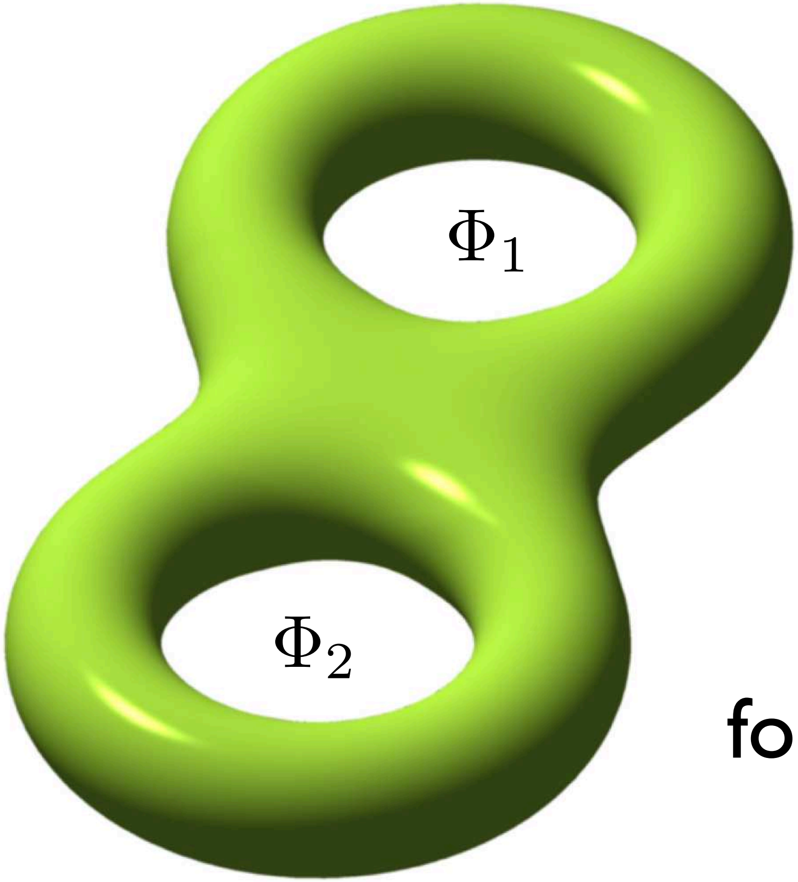
adiabatic flux insertion then eliminate the flux by the large gauge tr.

changes the eigenvalue of $\mathcal{T}_x(0)$!!

You come back to a different ground state

⇒ ground-state degeneracy (at least *q*-fold)

Topological Order



system on 2d manifold with genus *g* (*g* "holes")

we can apply the flux-insertion argument for each "hole"

$$\text{ground-state degeneracy} \geq q^g$$

for bosons/fermions

$$\text{ground-state degeneracy} \geq q^{2g}$$

Ground-state degeneracy which depends on the topology

"topological degeneracy"

— signature of the "topological order"

Fractionalization requires topological order

Summary

Aharonov-Bohm effect:
important also for many-body systems

Unit flux quantum is equivalent to zero flux
with respect to the AB effect: the equivalence is
shown explicitly by the "large gauge transformation"

Adiabatic insertion of the unit flux quantum in
a gapped system: should bring back a ground state
to a ground state, but under certain conditions,
the final state cannot be identical to the initial state

Example: system with a fractional charge

$\Rightarrow$ topological ground-state degeneracy