I work in algebraic topology, specifically stable homotopy theory. Some of the relevant keywords are model categories, spectra, Bousfield localisation, A_∞-algebras, homological algebra and homotopical algebra. If you like the idea of working on the algebraic side of algebraic topology, here is a brief overview of the topics I can offer projects in, including some further reading suggestions. If you have any questions or would like to hear more, please drop me an email!

1 Model categories and rigidity

Homotopy

A central notion in topology is that of homotopy. Two maps $f, g : X \longrightarrow Y$ of topological spaces are *homotopic* if there is a homotopy between them, i.e. a continuous map $H : [0, 1] \times X \longrightarrow Y$ such that $H(0, -) = f$ and $H(1, -) = g$. We can think of the homotopy as a way of deforming f into g continuously in finite time.

On the other hand, in algebra there is also the definition of *chain homotopy*. Two chain maps of chain complexes $f, g : (C_*, d_C) \longrightarrow (D_*, d_D)$ are said to be *chain homotopic* if there are maps $H_n : C_n \longrightarrow D_{n+1}$ such that $f_n - g_n = d_D \circ H_n + H_{n-1} \circ d_C$. But why does this deserve the name "homotopy"? Where is the unit interval or the continuous deformation? The world of model categories tells us that these two definitions do not just "morally" agree somehow but that they are in fact special cases of a more generalised notion of homotopy.

Model categories and a more general notion of homotopy

Roughly speaking, a *model category* is a category with a sensible notion of homotopy between morphisms. In a model category, there are distinguished classes of morphisms called fibrations, cofibrations and weak equivalences satisfying a strong but natural set of axioms involving retracts, factorisation and lifting properties. You might have heard of the definition of fibrations and cofibrations in topology- in fact, the model category definition says that those generalised fibrations and cofibrations are supposed to satisfy exactly the conditions of e.g. Serre fibrations and cofibrations in topology. In this example of topological spaces, the weak equivalences play the role of the weak homotopy equivalences.

Those axioms let us define what homotopy between morphisms is without having to have a "unit interval" in our category. Examples of categories with such a model structure involve, as mentioned earlier, topological spaces and chain complexes. In chain complexes, the weak equivalences are the quasi-isomorphisms, i.e. those chain maps that induce an isomorphism in homology.

The homotopy category

With a model category $\mathcal{C}$, one can form its *homotopy category* $Ho(\mathcal{C})$. Its objects are the objects of $\mathcal{C}$ but its morphisms are the homotopy classes of morphisms in $\mathcal{C}$, using the notion of homotopy derived from the model category axioms. So a special case would be the category of topological spaces together with homotopy classes of continuous maps. The homotopy category might be in some ways easier to study than the original underlying model category, but one loses information when passing to homotopy level.

Uniqueness and other open questions

Here is where the question of *rigidity* can be asked: how much of the underlying model structure can be recovered from just the structure of the homotopy category alone? Are there seemingly different categories which model the same homotopy category? And if yes, how much of their "higher homotopy structure" still agrees and why? Usually, algebraic model categories behave different from the topological example, but how exactly?

There are some interesting examples which have been studied in the last decade, but these are still rare and sometimes mysterious. In the future, hopefully more can be discovered about "rigidity", algebraic models, exotic models, uniqueness of underlying model structures and their relations. A PhD project in this area of algebraic topology could contribute to this fascinating topic.

Reading

Introductory:

W.G. Dwyer and J. Spalinski: Homotopy theories and model categories. Handbook of algebraic topology, 73126, North-Holland, Amsterdam, 1995.

More advanced:

B. Shipley: Rigidity and algebraic models for rational equivariant stable homotopy theory (talk notes), 2011.

`http://homepages.math.uic.edu/ bshipley/banff.17.6.pdf`

Research article:

S. Schwede: The stable homotopy category has a unique model at the prime 2, Advances in Mathematics, 164(1), 24–40, 2001.

2 A_∞-algebras

The basics

An *associative algebra* is a module over a ring (e.g. a vector space) A that is equipped with an associative multiplication $\mu : A \otimes A \longrightarrow A$. (Usually, we just write $x \cdot y$ for $\mu(x \otimes y)$.) Standard examples of such associative algebras include the polynomial algebra $R[X]$ over a ring R or the algebra of square matrices over a ring.

Because we ask for the multiplication to be associative, there is only one way of multiplying n elements $x_1, ..., x_n$ of A as the associativity tells us that the "bracketing" of the product does not matter. An A_∞-algebra is a kind of algebra where the bracketing *does* matter. To be more precise, for each number n there is a multiplication map

$$m_n : A \otimes ... \otimes A \longrightarrow A$$

which says how to multiply n elements. Furthermore, when composed those multiplication maps have to satisfy the relation

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \tag{1}$$

for all possible numbers of input n. As an exercise you can try and work out that an associative algebra is an A_∞-algebra with $m_i = 0$ for $i \neq 2$. (Hint: in that case, the above relation reduces to the associativity condition.) Similarly, a chain complex is an A_∞-algebra with $m_i = 0$ for $i \neq 1$ and a differential graded algebra is an A_∞-algebra with $m_i = 0$ for $i \neq 1, 2$.

A_∞-algebras and loop spaces

So the definition of an A_∞-algebra generalises the definition of an associative algebra. While the above relation looks quite random at first glance, it in fact has some strong geometric roots. One of the motivations for this definition is to describe the loop space ΩX of a topological space X in an algebraic way. (More specifically, the singular chain complex of a loop space is an A_∞-algebra.)

If X is a topological space with distinguished base point $*$, then its loop space is defined as the set of all closed loops in X

$$\Omega X = \{\gamma : [0, 1] \longrightarrow X \mid \gamma(0) = \gamma(1) = *\}.$$

(This is not just a set- it becomes a topological space via the compact-open topology.) This space *almost* has a multiplication map $\odot : \Omega X \times \Omega X \longrightarrow \Omega X$ given by the concatenation of loops. Why almost? If you concatenate two loops of length 1, the result is a loop of length 2- you have to rescale the result to get a loop of length 1 again by going through each of the two parts at twice the speed. This "product" $\odot$ is not associative: this rescaling means that $((\gamma_1 \odot \gamma_2) \odot \gamma_3)$ and $(\gamma_1 \odot (\gamma_2 \odot \gamma_3))$ do not strictly agree but agree *up to homotopy.* This is a similar scenario to the relation of the m_n in the definition of an A_∞-algebra: $m_2(1 \otimes m_2)$ and $m_2(m_2 \otimes 1)$ may not strictly agree but they agree up to some relation.

Derived A_∞-algebras

For many useful results regarding A_∞-algebras, one needs the assumption that A is not just a module over a commutative ring but a vector space over a field. (Or, more weakly, one needs the A_∞-algebras to be "projective", which is automatically the case if one works over a base field rather than a ring.) This is rather awkward as many naturally occurring examples in topology require the base ring to be $\mathbb{Z}$ or the p-local integers $\mathbb{Z}_{(p)}$. To bypass these assumptions, *derived A_∞-algebras* have been developed recently.

The idea is that if an A_∞-algebra is not projective, one can build in a projective resolution that is compatible with the A_∞-relation. This roughly means that if an object is not "nice" enough, then one can replace it with another object that is sufficiently nice but will be much bigger. This is why we do not give the definition of a derived A_∞-algebra here but just say that it is a formula much bigger than (1)!

Open questions

Because this formula is rather big, it can be hard to calculate actual examples with it. Also, it would be interesting to find out more about how this structure of derived A_∞-algebras is related to other already well-known structures. For example, A_∞-algebras and associative algebras are closely related: A_∞-algebras can be thought of a “resolution” of associative algebras in the above sense- a relatively small formula is replaced by a bigger one that has nicer properties. Is something similar true for derived A_∞-algebras- i.e. is there a smaller, well-known structure that relates to them like this? Can one also “derive” other structures, e.g. Lie algebras, to obtain a technique that is applicable to more examples?

This field of “algebraic operads” (e.g. the study of algebraic structures and their relation to each other) has gained a lot of momentum in the last decade because of its relations to algebra, topology and geometry as well as modern category theory. Hence, there are many open questions relating to existing results that can lead to an interesting project.

Reading

Introductory:

B. Keller: Introduction to A-infinity algebras and modules. Homology, Homotopy and Applications, 3(1), 1–35, 2001.

More advanced:

J.E. McClure and J. Smith: Operads and cosimplicial objects: an introduction. In: Axiomatic, enriched and motivic homotopy theory, 2004.

Research article:

M. Livernet, C. Roitzheim and S. Whitehouse: Derived A_∞-algebras in an operadic context, Algebraic and Geometric Topology, 13(1), 409–440, 2013.

C. Roitzheim, 2013