An Improved Method for Pricing and Hedging American Options

Tommaso Paletta
Silvia Stanescu
Radu Tunaru
Kent Business School
An Improved Method for Pricing and Hedging American Options

Tommaso Paletta\textsuperscript{a}, Silvia Stanescu\textsuperscript{a}, Radu Tunaru\textsuperscript{a,*}

\textsuperscript{a}Business School, University of Kent, Park Wood Road, Canterbury CT2 7PE, UK

Abstract

The majority of quasi-analytic pricing methods for American options are efficient near-maturity but are prone to larger errors when time-to-maturity increases. A new methodology, called the 'extension'-method, is introduced to increase the accuracy of almost any existing quasi-analytic approach in pricing long-maturity American options. It relies on an approximation of the optimal exercise price near the beginning of the contract combined with existing pricing approaches so that the maturity range for which small errors are attainable is extended. The new methodology retains the quasi-analytic nature of the methods it improves on and we derive generic quasi-analytic formulae for the price of an American put as well as for its delta parameter. Our numerical study indicates that the proposed methodology considerably improves both the pricing and the hedging performance of a number of established approaches for a wide range of maturities. Furthermore, the pricing improvements are most sizeable at longer maturities, where existing approaches do not perform well.

Keywords: American options, Optimal exercise price, Quasi-analytic method, Delta-Hedging performance

\textit{JEL:} G13, C63

\textsuperscript{*}Corresponding author. Tel: +44(0)1227 824608, Fax: +44 (0)1227 827726
Business School, University of Kent, Park Wood Road, Canterbury CT2 7PE, UK
Email address: r.tunaru@kent.ac.uk (Radu Tunaru)
1. Introduction

The problem of pricing American options has been widely examined in the last 40 years, but still there is no exact solution, not even under the Black-Scholes model. The main challenge consists in the fact that the American optionality requires the selection of the optimal exercise strategy (OEP) together with the valuation of the contingent claim. Several types of approximation approaches have been proposed in the literature to solve this problem but there is no single method outperforming the others in terms of pricing error, computational cost, hedging performance, mathematical and/or programming complexity and accuracy in optimal exercise prices, see Li (2010a).

Recently the static hedging portfolio approach to price American options has also been used successfully for equity, see Chung and Shih (2009), defaultable equity as in Ruas et al. (2013) and for barrier options Chung et al. (2013).

Within the broad class of approximation methods, we focus here on the quasi-analytic methods consisting of analytic formulae that require at most a reasonably small number of numerical solutions of (integral) equations. The first method in this subclass is described in Geske and Johnson (1984), henceforth GJ, who used a portfolio of compound European options to replicate the early exercise feature of American options. Ho et al. (1997) used a generalization of this method to price American options on bonds and Bunch and Johnson (1992) improved the efficiency of the GJ method by optimally locating the exercise points and showed that most of the time only two – and in few cases for deep-in-the-money options only three – early-exercise dates including maturity are required. A remarkable technique is the quadratic approximation in Barone-Adesi and Whaley (1987), henceforth BAW, that gives an approximated solution of the Black-Scholes PDE in closed form.
This method, extremely fast and accurate for very short and very long maturities, has been refined by Ju and Zhong (1999), henceforth JZ, including a second-order extension that improves accuracy for middle-term maturities. Subsequently, Li (2010a) further refined JZ by improving the accuracy of the OEP estimation. However, the approximations in BAW, JZ and Li (2010a) have the limitation that the error cannot be controlled. Moreover, Omberg (1987) approximated the OEP with an exponential function and Ju (1998) proposed a piece-wise exponential function for the OEP.

An important step in the American option pricing literature was the result of Kim (1990), henceforth K, who derived an implicit-form integral equation for the OEP \(^1\). Hence, the pricing of American options can be reduced to identifying the OEP efficiently. Several subsequent papers focused on improving the computational performance of the integral method. Among them, Sullivan (2000) employed Chebyshev polynomials and Gaussian quadrature, Kallast and Kivinukk (2003) used the trapezoidal rule and the Newton-Raphson method, and Kim et al. (2013), based on an idea from Little et al. (2000), transformed the integral equation into a numerical functional form with respect to the optimal exercise boundary, and subsequently constructed an iterative method to calculate the boundary as a fixed point of the functional. Moreover, Carr (1998) priced American options from a series of random maturity options but this method appears to be quite slow as pointed out by Sullivan (2000).

Broadie and Detemple (1996) calculated tight lower and upper bounds for the American call option prices under the assumption of a constant OEP.

\(^1\)Further technical details can be found in Carr et al. (1992).
Chung et al. (2010) improved on these bounds under the assumption of an exponential OEP. Another fast approach is described by Li (2010b) (henceforth LI) who, following the regression approach in Johnson (1983), derived a closed-form pricing formula by exploring two bounds of the American option price. Finally, Chung and Shih (2009), henceforth CS, proposed a static replicating portfolio of European options with different strikes and maturities in order to price American options. Reviews on American option pricing are in Broadie and Detemple (1996), Barone-Adesi (2005) and Pressacco et al. (2008).

All the methods above share the problem that they may produce large pricing errors for long-maturity options since, for most of them, the convergence depends on the decrease of the size of the time-step or, equivalently, on the increase of the number of iterations. However, an increase in the number of iterations makes these methods rapidly inefficient. The majority of these methods resorts to extrapolation techniques, principally to Richardson’s extrapolation and consequently the computational time is still high. In Table 1 – see rows ‘S’, which contain the results for the ‘standard’ as opposed to ‘extended’ version of the methods – the performance of several pricing methods is depicted with respect to the mean absolute percentage error, MAPE, for maturities ranging from less than 6 months to between 4.5 and 5 years. All the methods considered perform well for short maturities. Several studies report similar findings.

In the paper, the word ‘standard’ is meant to be opposite to ‘extended’. The former refers to each method as described in the literature and as reported in Section 4. However the latter is given as in formulae (4) and (5), for pricing and hedging purposes respectively.

The comparison is done by using: Figures 3 and 4 in Broadie and Detemple (1996, p. 1227), Tables 2a-2e and 3a-3e in Ait-Sahalia and Carr (1997, pp. 76-85) (the results are
In the options market it is common to find options with maturities of up to 2-5 years, so the problem of how to improve the performance of these methods is relevant from both an academic and a practitioners’ point of view. We propose here a quasi-analytic approach that aims to improve the performance of existing methods in pricing long-maturity options: see Table 1, rows ‘E’ (for ‘extended’), where it is shown that our proposed ‘extension’ methodology always outperforms existing pricing approaches, across all maturity ranges considered, and that pricing improvements are most sizeable for longer maturities, where existing methodologies tend to perform worst. Since the new approach extends the ranges of maturities for which an existing quasi-analytic method returns good results, we call it the ‘extension’-method.

The proposed method resorts to two properties of the optimal exercise price (OEP) that, to the best of our knowledge, have not been used together before for pricing purposes: firstly, the observation that the OEP is almost constant for the first part of the option life, and secondly, the fact that the OEP is independent of the current underlying asset price. In particular, we divide each option’s time-to-maturity in two components according to the closeness to the maturity date and we use a constant approximation function for the first part of the option life and existing pricing methods (with their associated estimation approaches for the OEPs) for the second part. Consequently, under the proposed ‘extension’ methodology, the option price is equal to the sum of the expected discounted-payoff from the first part of the option life and the expected discounted-payoff from the second part.

for options on a non-dividend paying asset for short time-to-maturity and on a dividend paying asset for long time-to-maturity options). Exhibits 3 and 5 in [Ju and Zhong (1999), Tables 3, 4 and 5 in Li (2010b, pp. 91-93), Figures 4 and 5 in Kallast and Kivimukk (2003, pp. 373-374) and Tables 4 and 5 in Kim et al. (2013, p. 7).
part, conditioned on not exercising the option in the first part. An equivalent formula is provided for the Delta parameter. Our method is a generalization of Bjerksund and Stensland (2002) where the life of the option is divided in two parts and for both of them a flat approximation of the OEP is employed; it has the same rationale but, unlike Bjerksund and Stensland (2002), our method works with a wide range of pricing formulae.

Furthermore, we also show that our method can be efficiently used together with asymptotic approximations of the OEP near expirations in order to price long-maturity options. Consequently, we show that the asymptotic approximations, which according to Barone-Adesi (2005) and Chung and Shih (2009) constitute one of the main research streams in quasi-analytic methods for American-options, are complementary not only to numerical methods (in particular binomial tree) as suggested in Evans et al. (2002) but also to quasi-analytic methods.

Traditionally, the approximations of the OEP near expiration have been based on perturbation or asymptotic methods. Among them Kuske and Keller (1998) derived an asymptotic formula similar to Barles et al. (1995) based on the integral equation in Kim (1990) for non-dividend stock options. Evans et al. (2002), henceforth EKK, proved the asymptotic expansion of the OEP for any value of the dividend yield, as a generalization of Kuske and Keller (1998). Zhang and Li (2010), henceforth ZL, generalized Evans et al. (2002) considering the first four terms of the expansion by using the perturbation methodology in Chen and Chadam (2007). Finally, Chung et al. (2011) found results consistent with Barles et al. (1995) by exploring a relationship between the OEP and the gamma of the American put on non-dividend paying stock.
All these traditional methods are designed for a short time-to-maturity of at most a few months. Cheng and Zhang (2012), henceforth CZ, provided an explicit approximate formula for the OEP function that is valid for a time-to-maturity of a few years and that covers the case of dividends (for a dividend yield lower than the risk-free rate).

To summarize our contribution, this paper extends previous research on American options in several ways. Firstly, we propose a new quasi-analytic approach for pricing and hedging American options. This relies on an approximation of the OEP – constructed based on its documented theoretical properties – in order to extend the applicability of established quasi-analytic methods, which are successful in pricing short maturity options, to longer maturity options. In the context of the proposed framework, we derive formulae for the American put price and for the corresponding delta. We also show the convergence of the put price obtained with our proposed (‘extension’) method to the perpetual put price, when maturity increases infinitely. On the numerical side, we provide an extensive study which shows that, when compared with established quasi-analytic methods, the proposed approach leads to sizeable improvements in both pricing and hedging American options, especially at longer maturities where existing methods generally fail.

The remainder of the paper is organized as follows. Section 2 describes the modelling framework. The main theoretical results are discussed in Section 3 where the closed-form pricing and hedging formulae are derived. Section 4 briefly describes some quasi-analytic pricing methods which can be easily ‘extended’ by the new method. Finally, Section 5 is a numerical evaluation of the pricing and hedging performance of the ‘extension’-method and Section 6 concludes.
2. Modelling framework

All modelling referring to American option pricing in this paper is done assuming the Black-Scholes model. Hence, under the risk-neutral measure $\mathbb{Q}$, the dynamics of the underlying stock $S$ are given by:

$$dS_t = (r - \delta)S_t dt + \sigma S_t d\tilde{W}_t, \quad t \geq t_0$$

(1)

where $r$ is the risk-free rate and $\delta$ is the annual dividend yield with continuous compounding. For simplicity the difference $r - \delta$ is denoted henceforth by $b$ and $\{\tilde{W}_t\}_{t \geq t_0}$ is a Wiener process under the martingale measure $\mathbb{Q}$.

Considering the put-call symmetry proved by McDonald and Schroder (1998), the problem of pricing American call options can be reduced to that of pricing American puts. Consequently, without any loss of generality, we only consider the case of American put options below. We next review a number of documented properties of the OEP of an American put which will be useful in deriving some of the theoretical results in this paper, as well as motivating our methodology. The optimal exercise price of the American put option with maturity $T$ and strike price $K$ is a continuous function, see Jacka (1991), non-decreasing with respect to time, bounded above by $\min\{K, rK/\delta\}$ when dividends are paid at the rate $\delta$, and below by the

---

4Let $P_t(S_t, K, r, \delta, T)$ and $C_t(S_t, K, r, \delta, T)$ denote the time-$t$ price functions of American put and call option respectively, with identical characteristics, and $S_f(K, r, \delta, T, t)$ and $S'_f(K, r, \delta, T, t)$ the corresponding optimal exercise prices, then $P_t(S_t, K, r, \delta, T) = C_t(K, S_t, \delta, r, T)$ and $S_f(K, r, \delta, T, t) = \frac{S_tK}{S'_f(S_t, \delta, r, T, t)}$. 

---

8
optimal exercise price of the perpetual put option, \( S^\infty_f \), where

\[
S^\infty_f = \frac{\beta}{\beta - 1} K, \tag{2}
\]

with \( \beta = \left( \frac{1}{2} - \frac{b}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{b}{\sigma^2} \right)^2 + 2 \frac{r}{\sigma^2}} \). Ekstrom \cite{2004} and Chen et al. \cite{2008} show that the OEP is convex when no dividends are paid and Chen et al. \cite{2013} prove that for \( 0 < \delta - r \ll 1 \) the OEP loses convexity near maturity. Chen et al. \cite{2011} for dividend paying options and Xie et al. \cite{2011} for the non-paying dividend case showed that the decay rate of the OEP to the flat function in (2) is more than exponential in time-to-maturity.

From the results above, it seems advantageous to focus any computational effort in the estimation of the OEP close-to-maturity since it changes slope rapidly and it can be difficult to estimate and use an approximation of the OEP far from maturity. Consequently, the key to our approach is to divide the option life in two parts, one closest to the beginning of the contract and one to maturity, and use existing pricing methods and their corresponding estimation approaches for the OEP in the second part while considering a flat approximation of the OEP for the first part. In doing so the advantage is threefold: the estimation of the OEP near maturity is precise, the existing methods are used where they have better performance (comparative advantage), and very low computational effort is required near the beginning of the contract where the theory suggests that the OEP is ‘nicer’. The ‘extension’-method is based on the well-known property discussed in Geske and Johnson \cite{1984}, Kim \cite{1990} and Basso et al. \cite{2004} that, under the Black-Scholes model, the optimal exercise price does not depend on the current spot price. As a consequence, it is possible to employ the OEP of a shorter
maturity option to build part of the optimal exercise price of an American option written on the same asset, with the same strike price but with longer maturity.

Figure 1 depicts the intuition behind our method. For a given set of parameter values\(^5\), the figure plots the OEPs for two American put options, written on the same underlying asset, with maturities \(t_1\) and \(T\), where \(t_1 < T\). Assume that we are now at time \(t_0\) and consider the intermediary time point \(t_x = t_0 + (T - t_1)\).\(^6\) Because the two options have identical characteristics apart from their different maturity dates, and because the OEP does not depend on the prevailing spot price, the optimal exercise prices for the two options will coincide whenever the options have the same time to maturity. In particular, for any time \(t\) in the interval \([t_x, T]\), the OEP of the long maturity option will be the same as the short maturity option’s OEP, which is defined on \([t_0, t_1]\). In the figure, the continuous line represents the optimal exercise price of the option with maturity \(T\) and the dash dot lines represent the optimal exercise of the option with maturity \(t_1\). The left-most dash dot line is the ‘original’ function and the other is its translation over the continuous line to show they coincide on the interval \([t_x, T]\).\(^7\)

3. The ‘extension’-method

We describe here the ‘extension’-method which, as stated above, can be employed together with any quasi-analytic pricing method to improve the accuracy of the latter especially for long maturity options. The first step is

\(^5\)\(\sigma = 20\%, \delta = 5\%, r = 8\%\) and \(K = 100\).

\(^6\)In Figure 1 we assume \(t_0 = 0, t_1 = 1\) year, \(T = 2.5\) years and consequently \(t_x = 1.5\) years.

\(^7\)The OEPs are calculated by the integral method in Kim (1990).
Figure 1: Example of Extension Method mechanism.
The optimal exercise prices of two American put options are considered in the picture. The two options are written on the same underlying asset with $\sigma = 20\%$, $\delta = 5\%$, $r = 8\%$ and $K = 100$. Furthermore, one has maturity $t_1 = 1$ year and the other $T = 2.5$ years. The continuous line represents the optimal exercise price of the option with maturity $T$ and the dash dot lines represent the optimal exercise of the option with maturity $t_1$. In particular, the left-most dash dot line is the ‘original’ function and the other is its translation over the continuous line to show they coincide in the interval $[t_x, T]$ where $t_x = t_0 + (T - t_1) = 1.5$ years represents the size of the translation. The OEPs are calculated by the integral method in [Kim (1990)].

to split the option life in two parts: for the first part (i.e. the one closest to the beginning of the contract), we approximate the OEP as a constant $X$, while the pricing method we ‘extend’ provides the OEP and the pricing formula for the part closest to maturity. Consequently, the assumption made here is that the optimal exercise price $S_j^{(E)}(\cdot)$ of an American put option,
with maturity $T$, starting life at $t_0$, is given by:

$$S_f^E(t) = \begin{cases} X & \text{for } t \in [t_0, t_x) \\ S_f(t - (T - t_1)) & \text{for } t \in [t_x, T] \end{cases}$$

(3)

where $t_x \in [t_0, T]$ is the break-point (i.e. the time point separating the option life in two parts, as explained above). $S_f(\cdot)$ is the optimal exercise price of the shorter maturity option (written on the same underlying asset, with the same strike price). As explained in Figure 1 we can think of the shorter maturity option as either starting life at $t_0$ and having maturity date $t_1$, or as starting life at $t_x$ and having maturity date $T$. In either case, at the onset, the shorter maturity option has $t_1 - t_0 = T - t_x$ time (years) to maturity. $S_f(\cdot)$ will be estimated via an existing (standard) quasi-analytic method – e.g. one of the methods described in Section 4 – which we are extending to price the long maturity option.

With the OEP given in (3), the price of the American put option is calculated as the sum of the expected discounted payoff (between $t_0$ and $t_x$), assuming that the option is exercised as soon as the spot price hits $X$, and the expected discounted payoff from the short maturity American option (between $t_x$ and $T$) conditioned on not hitting $X$ between $t_0$ and $t_x$. Proposition 3.1 derives the pricing formula of the ‘extension’-method, where the

---

We note that selecting $t_x \to t_0$, the option price obtained with our proposed ‘extension’-method converges to the price obtained via the method we are extending (i.e. what we call the corresponding ‘standard’ method) and when $t_x \to T$, the ‘extension’-method price converges to the price of the method in Bjerksund and Stensland (1993). The numerical study in Section 5 shows that for intermediate values of $t_x$, each ‘extended’ method provides better prices (according to a number of criteria) than the corresponding ‘standard’ version. Figure 2 plots for each of the methods in Section 4 the Mean absolute Percentage Error (MAPE) as a function of $t_x$. 

12
following notation will be used: $P_{t_x}(S_{t_x}, T, K)$ is the price of the option with time to maturity $T - t_x$ (short maturity option) at time $t_x$ when the underlying asset price is $S_{t_x}$ and the OEP is given by $S_{f_x}(\cdot)$ -- a function defined on $[t_x, T]$, which is the translation of the function $S_f(\cdot)$, the latter being defined on $[t_0, t_1]$ -- and $P_{t_0}^{(E)}(S_{t_0}, T, K|t_x, X)$ is the price of the option with time to maturity $T - t_0$ (long-maturity option), at time $t_0$, when the underlying asset price is $S_{t_0}$ and when the optimal exercise price is given by (3).

We also use the simplified notation $\varphi(\gamma, H)$ to denote the expectation term

$$
\varphi_t(S_{t_0}, t_x|\gamma, H, X) = E_{t_0} \left[ e^{-r_{t_x}S_{t_x}} I (S_{t_x} > H) I (\inf_{t \in [t_0, t_x]} S_t > X) \right],
$$

which, for reasons of space, is only given in Appendix A (see equation A.3), together with its derivation and also with the expression for $f_0(\cdot)$ (see equation A.5) -- the probability density function of an arithmetic Brownian motion with positive initial value $z_{t_0}$, drift parameter $b_1 = b - \frac{1}{2}\sigma^2$, volatility parameter $\sigma$ and an absorbing barrier at 0.

**Proposition 3.1.** Assuming Black and Scholes dynamics, the price of an American put option with strike price $K$ and maturity $T$ at time $t_0$, based on the ‘extension’ of the method with pricing function $P_{t_x}(S_{t_x}, T, K)$, is given by:

$$
P_{t_0}^{(E)}(S_{t_0}, T, K|t_x, X) = e^{r_{t_0}} \left\{ \alpha(X) \left[ S_{t_0} e^{-r_{t_0}} - \varphi(\beta, X) \right] - \varphi(1, X) 
+ \varphi(1, S_{f_x}^{(E)}(t_x)) + K \left[ \varphi(0, X) - \varphi(0, S_{f_x}^{(E)}(t_x)) \right] \right\} 
+ \int_{B}^{+\infty} g(z) dz,
$$

where

$$
g(z) = e^{-r(t_x - t_0)} P_{t_x}(X e^{-z}, T, K) f_0(z),
$$

13
\[ B = \ln \frac{S_f(x)}{X}, \quad \alpha(X) = (K - X)X^\beta \quad \text{and} \quad \beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}}. \]

Proof. See Appendix A.

The proof of this proposition is similar to the proof for the pricing formula in Bjerksund and Stensland (2002). Thus, when the OEP for the short maturity option is flat, the method described in Bjerksund and Stensland (1993) applies and our pricing formula becomes the pricing formula in Bjerksund and Stensland (2002). Therefore, the ‘extension’-method can be seen as a generalization of Bjerksund and Stensland (2002) that combines any quasi-analytic pricing formula for the short-maturity American put option and a flat approximation of the OEP near the beginning of the contract.

In order to satisfy the continuity property of the OEP, the constant \( X \) is fixed to be equal to \( S_f(x) \). All the empirical results in Section 5 are obtained for this equality holding\(^9\) however other available choices are the initial guess in Barone-Adesi and Whaley (1987) or the one in Bjerksund and Stensland (1993). Furthermore, the following proposition shows that asymptotically our pricing formula converges to the perpetual put option price formula.

**Proposition 3.2.** For any \( t_x \in (t_0, T] \), any \( X \) and any pricing formula for

\(^9\)A similar method, called the lower bound approximation (‘LBA’) has been described by Broadie and Detemple (1996).

\(^{10}\)The results for the integral method and the compound-option method (both described in Section 4), are obtained for \( X \) equal to the OEP at time \( t_x \) calculated by the quadratic method in Barone-Adesi and Whaley (1987) (pricing formula 8) since the calculation of \( S_f(x) \) by any of the two methods is very poor when only few early/exercise dates are considered.
the short-maturity option $P_{t_x}(S_{t_x}, T, K)$, when $T \to \infty$, the price

$$P_{t_0}^{(E)}(S_{t_0}, T, K|t_x, X)$$

given in Proposition 3.1 converges to the price of a perpetual option written on the same underlying asset, with the same strike price and which is exercised as soon as the underlying asset price hits $X$.

A proof is given in the Appendix B.

In addition to option pricing, the calculation of the Delta parameter is equally important in financial markets. The following result provides an analytic formula for the calculation of the Delta parameter of an American put option by the ‘extension’-method, relying on the independence of the OEP from the current asset price.

**Proposition 3.3.** Under the same conditions as stated in Proposition 3.1, the Delta parameter is given by the following formula

$$\Delta_{t_0} = e^{r_{t_0}} \left\{ \alpha(X) \left[ \beta S_{t_0}^{\beta-1} e^{-r_{t_0}} - \varphi'(\beta, X) \right] - \varphi'(1, X) 
+ \varphi'(1, S_f^{(E)}(t_x)) + K \left[ \varphi'(0, X) - \varphi'(0, S_f^{(E)}(t_x)) \right] \right\} 
+ \int_B^{+\infty} g'(z)dz, \quad (5)$$

where

$$g'(z) = e^{-r(t_x-t_0)} P_{t_x}(X e^{-z}, T, K) f_{t_0}'(z) .$$

**Proof.** Here we use the simplified notation $\varphi'(\gamma, H)$ to denote the partial derivative $\varphi'_{t_0, S_{t_0}}(S_{t_0}, t_x | \gamma, H, X) = \frac{\partial \varphi'_{t_0}(S_{t_0}, t_x | \gamma, H, X)}{\partial S_{t_0}}$, which is given in Ap-
pendix C together with the expression for \( f'_0(z) = \frac{\partial f_0(z)}{\partial S_{t_0}} \). The result is an application of the Leibniz’s derivation formula to function (4) and of the results \( \frac{\partial B}{\partial S_{t_0}} = 0 \), \( \frac{\partial P_{tx}(X_{a-t}, T, K)}{\partial S_{t_0}} = 0 \), \( \frac{\partial X}{\partial S_{t_0}} = 0 \) which follow from the independence of the OEP from the current asset price.

The pricing formula in Proposition 3.1 and the Delta parameter in Proposition 3.3 work under any specification for the pricing formula of the short-maturity option. Choosing one or another pricing formula for this option only changes the last addend of formulae (4) and (5), i.e. the two integrals. The following section summarizes some quasi-analytic pricing methods which can be used together with the results in the propositions above.

4. Methods for short maturity option

For the sake of clarity, we briefly describe some well-known pricing methods for American put options that could be employed under our methodology. We consider quasi-analytic methods that do not depend on an optimization stage and/or parameters found by an intermediary regression step. For all the selected methods, as shown in Table 1, the pricing performance worsens when time-to-maturity increases. Furthermore, although most of these methods, in their original definition, include an extrapolation step (Richardson’s extrapolation), we shall not consider any extrapolations since we shall focus on the improvement of the method for a specific number of early-exercise dates.

Before the description of each method, let us enumerate some common factors underpinning the methodologies below. Firstly, we remind that \( P_{tx}(S_{t_x}, T, K) \) is the time-\( t_x \) price of an American option contingent on an underlying asset
with the dynamics specified in equation (1), with time-to-maturity \( \tau = T - t_x \), strike price \( K \) and with underlying price \( S_{t_x} \). Secondly, the OEP of the short-maturity option is calculated by solving

\[
K - S_{f_x}(t) = P_t(S_{f_x}(t), T, K), \forall t \in [t_x, T]. \tag{6}
\]

for \( S_{f_x}(t) \), when needed\(^{11}\).

We report only the formula for \( S_{t_x} > S_{f_x}(t_x) \), because when \( S_{t_x} \leq S_{f_x}(t_x) \), the option price is simply its immediate exercise \( P_t(S_{t_x}, T, K) = K - S_{t_x} \).

We shall not report the delta-parameter functions and they can be found in the references below and/or can be calculated by simple derivation rules. In the following, we will denote by \( \Delta_{t_x, T} = \frac{T-t_x}{2} \) the time-step size.

4.1. Quasi-analytic methods

**Compound-option Method (GJ)**

The pricing formula in Geske and Johnson \(1984\)\(^{12}\) when considering two steps, is\(^{13}\)

\[
P_{t_x}(S_{t_x}, T, K) = KU_2(S_{t_x}) - S_{t_x}W_2(S_{t_x}), \text{ for } S_{t_x} > S_{f_x}(t_x) \tag{7}
\]

\(^{11}\)Note that for some methods (Barone-Adesi and Whaley (1987), Ju and Zhong (1999) and Li (2010b)) only \( S_{f_x}(t_x) \) is needed while for others (Geske and Johnson (1984), Kim (1990) and Chung and Shih (2009)) also the calculation of \( S_{f_x}(\cdot) \) for intermediate times is required.

\(^{12}\)The correction for dividend paying stock in Prekopa and Szantai (2010) is applied.

\(^{13}\)We note that in Section 3 we apply the compound-option (GJ) methodology, as well as the integral (K) and static replication portfolio (CS) methodologies, with three as well as two steps. However, in the interest of brevity, we only present the formulae for two time steps in this Section.
where

\[
U_2(S) = e^{-r\Delta t_x, T} N (-d_2(S, q_1, \Delta t_x, T)) + e^{-2r\Delta t_x, T} N_2 \left( d_2(S, q_1, \Delta t_x, T), -d_2(S, K, 2\Delta t_x, T), -\frac{1}{\sqrt{2}} \right),
\]

\[
W_2(S) = e^{-\delta \Delta t_x, T} N (-d_1(S, q_1, \Delta t_x, T)) + e^{-2\delta \Delta t_x, T} N_2 \left( d_1(S, q_1, \Delta t_x, T), -d_1(S, K, 2\Delta t_x, T), \frac{1}{\sqrt{2}} \right),
\]

\[q_1 = S_{f_x}(t_x + \Delta t_x, T)\] and \[S_{f_x}(t_x)\] solve (6).

**Quadratic Methods (BAW and JZ)**

The pricing formula in Barone-Adesi and Whaley (1987) is:

\[
P_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) + A_1 \left( \frac{S_{t_x}}{S_{f_x}(t_x)} \right)^{\beta_1} \text{ for } S_{t_x} > S_{f_x}(t_x)
\]

(8)

where

\[
A_1 = -\frac{S_{f_x}(t_x)}{\beta_1} \left[ 1 - e^{(b-r)T} N (-d_1(S_{f_x}(t_x))) \right], \quad \beta_1 = \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left( \frac{1}{2} - \frac{b}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2 h}}
\]

\[h = 1 - e^{-rT} \text{ and } S_{f_x}(t_x)\] solves (6).

Ju and Zhong (1999) improve the approximation in Barone-Adesi and Whaley (1987) as follows:

\[
P_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) + \frac{A_2(S_{f_x}(t_x)) \left( \frac{S_{t_x}}{S_{f_x}(t_x)} \right)^{\lambda_1}}{1 - \chi} \text{ for } S_{t_x} > S_{f_x}(t_x)
\]

(9)

\[14\text{ The method in Li (2010a), a further modification of the quadratic method, is not considered here since it has almost the same pricing performance of JZ and is more computationally intensive.}\]
where the formula for $A_2(S_{f_x}(t_x))$ can be found in Ju and Zhong (1999) and it is not given here for lack of space. Furthermore, $S_{f_x}(t_x)$ solves

$$1 - e^{-\delta \tau} N(-d_1(S_{f_x}(t_x))) + \frac{\lambda_1 A_2(S_{f_x}(t_x))}{S_{f_x}(t_x)} = 0.$$ 

**Integral Method (K)**

The pricing formula in Kim (1990), Jacka (1991) and Carr et al. (1992) is:

$$P_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) +$$

$$+ \int_0^\tau \left[ r K e^{-r(\tau-t)} N(-d_2(S_{t_x}, S_{f_x}(t_x + \tau - t), \tau - t)) +$$

$$- \delta S_{t_x} e^{-\delta(\tau-t)} N(-d_1(S_{t_x}, S_{f_x}(t_x + \tau - t), \tau - t)) \right] dt, \text{ for } S_{t_x} > S_{f_x}(t_x)$$

(10)

where $S_{f_x}(t)$ solves (6). For implementation purposes, formula (10) is discretized: when 2 time-steps of equal length $\Delta_{t_x,T}$ are considered, formula

---

15We shall consider the basic integral method only and not any of the improvements in Sullivan (2000), Kallast and Kivinukk (2003) and Kim et al. (2013), among others, since only few iterations (early exercise dates) will be considered in the computational comparison in Section 5 and, consequently, the computational improvements do not have a big impact. However, it is worth pointing that it is also possible to ‘extend’ these improved methods.
becomes:

\[
P_t(S_t,T,K) = p_t(S_t,T,K) + \Delta_{t,T} \sum_{j=1}^{2} \left[ rK e^{-r\Delta_l^j} N \left( -d_2(S_t, S_f(t_x) + \Delta_l^j), \Delta_l^j \right) - \delta S_t e^{-\delta \Delta_l^j} N \left( -d_1(S_t, S_f(t_x) + \Delta_l^j), \Delta_l^j \right) \right]
\]

where \( \Delta_l^j = \Delta_{t,T}(2 - j) \).

**Interpolation method (LI)**

The pricing formula in Li (2010b) is

\[
P_t(S_t, T, K) = A_3 p_t(S_t, T, Ke^{\tau_T}) + (1 - A_3) p_t(S_t, T, K), \text{ for } S_t > S_f(t_x)
\]

where

\[
A_3 = A_4 \left( \frac{S_t}{S_f(t_x)} \right)^{q(b, \frac{r}{\Phi})}, q \left( b, \frac{r}{\Phi} \right) = \frac{1}{2} - \frac{b}{\sigma^2} - \frac{1}{2\sigma^2} \sqrt{(\sigma^2 - 2b)^2 + 8\frac{r}{\sigma^2}}
\]

\[
A_4 = \frac{e^{\delta \tau} - N(-d_1(S_f(t_x), K))}{[N(-d_1(S_f(t_x), Ke^{\tau_T})) - N(-d_1(S_f(t_x), K))] - \frac{q(b, \frac{r}{\Phi})D(S_t, \tau, K) S_f(t_x)e^{-r\tau}}{S_f(t_x)e^{-r\tau}}}
\]

\[
\Phi = 1 - e^{-r\tau}, D(S_t, \tau, K) = p_t(S_t, T, Ke^{\tau_T}) - p_t(S_t, T, K).
\]

**Static-replicating portfolio**

The pricing method in Chung and Shih (2009), considering two time-steps,
is

\[ P_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) + Z_2(S_{t_x}, T, K) \text{ for } S_{t_x} > S_{f_x}(t_x) \]  \hspace{1cm} (13)

where

\[ Z_2(S_{t_x}, T, K) = w_1p_{t_x}(S_{t_x}, T, S_{f_x}(t_x + \Delta_{t_x}, T)) + w_0p_{t_x}(S_{t_x}, T, S_{f_x}(t_x)) \]  \hspace{1cm} (14)

and \( w_1, S_{f_x}(t_x + \Delta_{t_x}, T) \) and \( w_0, S_{f_x}(t_x) \) backwardly determined as the solution to the smooth-pasting condition and the value matching condition in (6).

4.2. Near-maturity asymptotic expansions

Evans et al. (2002), EKK, arrived at the following approximation for the OEP near maturity\(^{16}\)

\[ S_{f_x}(t) \approx \begin{cases} 
K - K\sigma\sqrt{(T - t) \ln \frac{\sigma^2}{8\pi(T - t)(r - \delta)^2}} & \text{if } 0 \leq \delta < r \\
K - K\sigma\sqrt{2(T - t) \ln \frac{1}{4\pi\delta(T - t)}} & \text{if } \delta = r \\
\frac{7}{5}K \left(1 - \sigma\alpha_0 \sqrt{2(T - t)}\right) & \text{if } \delta > r
\end{cases} \]  \hspace{1cm} (15)

where \( \alpha_0 \approx 0.4517. \)

\(^{16}\)Among the several asymptotic expansions, only the contributions which consider dividend-paying stock are included.
Zhang and Li (2010), ZL, proposed:

\[
S_{fx}(t) \approx \begin{cases} 
  Ke^{-\sqrt{2\sigma^2(T-t)}u(\xi)} & \text{if } 0 \leq \delta < r \\
  Ke^{-\sqrt{2\sigma^2(T-t)}v(\eta)} & \text{if } \delta = r \\
  \frac{r}{\delta}Ke^{-2\sqrt{\tau^*}w(\sqrt{\tau^*})} & \text{if } \delta > r 
\end{cases}
\]

(16)

where the formulae for the terms \(u(\xi), v(\eta), w(\cdot)\) and \(\tau^*\) can be found in Zhang and Li (2010).

Finally, Cheng and Zhang (2012) provided an analytic approximation formula that is in essence an expansion in terms of powers of \(\sqrt{\frac{1}{2}\sigma^2(T-t)}\). The formula is rather long and it will not be reproduced here for lack of space. They provide three methods, the simple analytic formula calculated by the homotopy analysis method (henceforth CZ) in their formula [8], the expansion of CZ calculated by Pade’ approximation (henceforth CZ-P) as in their Appendix [C] and finally, CZ-P corrected by formula (15) for \(\delta > r\) (henceforth CZ-P-m).

These five near-maturity-asymptotic-expansion methods for the OEP will be used in combination with formula (10) to price the short-maturity options.

5. Numerical study

The aim of this section is to show the usefulness of the ‘extension’-method and, to this end, we will apply it to all methods reviewed in Section 4.1 comparing the performance of each standard method with its ‘extended’ version and trying to highlight that the performance improves considerably. Furthermore, we will present an analysis of the ‘extension’ of near-maturity asymptotic expansions of the OEP (Section 4.2) and their performance comparison. The focus will be on the accuracy since the computational effort required by
the ‘extension’-method is only slightly higher than the standard methods and, in most cases, the additional computational time is negligible.\textsuperscript{17}

The numerical study will be carried out for a flat approximation $X$ fixed equal to $S_{f_x}(t_x)$ in order to ensure the continuity of the OEP. However, for the integral method and the compound-option method, because when few steps are considered the value of $S_{f_x}(t_x)$ can be quite unreliable, $X$ is fixed to be equal to the OEP at time $t_x$ calculated by the quadratic method in Barone-Adesi and Whaley (1987). However, other approximations could have been used: for example similar results are obtained by the approximation in Bjerkund and Stensland (1993) which has the advantage to circumvent having to solve equation (6). Also for the asymptotic methods, we fix $X$ to the OEP at time $t_x$ calculated by the quadratic method in Barone-Adesi and Whaley (1987), as we do for the integral method.\textsuperscript{18} Moreover, without loss of generality, we fix $t_0 = 0$. We consider 10 ratios $t_x/T$ (5\%, 10\%, ..., 95\%) to study the performance of the ‘extension’-method under different approximations of the OEP. However, for the methods based on an asymptotic expansion of the OEPs we consider one exercise date per day (365 days per year) and the maximal expansion considered is of two years.\textsuperscript{19} For the asymptotic expansion methods, the study is carried out for a series of values for $T - t_x$ at

\textsuperscript{17}For the methods developed by Barone-Adesi and Whaley (1987), Kim (1990), Chung and Shih (2009) the integrals in (4) and (5) can be calculated analytically while the other require a numerical calculation of the integrals that is much faster than the numerical solution of integral equations.

\textsuperscript{18}The methods CZ, CZ-P and CZ-P-m have virtually the same performances when the flat approximation $X$ is fixed equal to $S_{f_x}(t_x)$. However, the methods EKK and ZL perform badly under this approximation for long maturity options since they are valid up to few months maturities. Consequently, for consistency we report all the results for $X$ equal to $S_{f_x}(t_x)$ calculated by the quadratic method in Barone-Adesi and Whaley (1987).

\textsuperscript{19}This is the maturity period suggested by Cheng and Zhang (2012).

\textsuperscript{20}In this case, the difference $T - t_x$ has been considered since the feasibility of the
intervals of 2 weeks. Overall 52 points were considered (2 weeks, 4 weeks, \ldots, 2 years). For this reason, options with maturities shorter than 2 weeks have been disregarded. Moreover, in order to satisfy the theoretical properties of the OEP, we disregarded an OEP-expansion in case of negative values, values above the strike price, non-monotonic function and non-real functions.

In the next two subsections, we will separately study the pricing and the hedging performance.

5.1. Pricing performance

The pricing-performance study is constructed from a total of 10,000 randomly generated scenarios. In particular, the parameters in (1) and the options characteristics are drawn as in Broadie and Detemple (1996): the volatility $\sigma$ is distributed uniformly between 0.1 and 0.6; the initial asset price $S_0$ is fixed at 100; the strike price $K$ is distributed uniformly between 70 and 130; the dividend rate $\delta$ is distributed uniformly between 0.0 and 0.10 with probability 0.8 and equal to 0.0 with probability 0.2; the risk-free interest rate is uniformly distributed between 0.0 and 0.1. Given the importance of time-to-maturity to establish the usefulness of the ‘extension’-method, we divide the simulated scenarios in 10 sets with equal cardinality. The sets (name in parentheses) have maturities in years in the ranges (0; 0.5] (A), (0.5; 1] (B), (1; 1.5] (C), (1.5; 2] (D), (2; 2.5] (E), (2.5; 3] (F), (3; 3.5] (G), (3.5; 4] (H), (4; 4.5] (I), (4.5; 5] (J). The exact fair price (benchmark) is the binomial tree price with 15,000 time steps. Options with prices less than 0.5 are disregarded.

We use three measures of error to compare each method with its ‘ex-
tended’ version: mean absolute percentage error (MAPE), number of best solutions found\textsuperscript{21} defined as number of scenarios for which the relative error of the extended method is smaller than the standard method, and maximum relative error. The performance of the methods is summarized in Tables 1 and 2 for quasi-analytic methods and Table 3 for asymptotic expansion methods.

For each ‘standard’ method considered, the ‘extension’-method increases its pricing performance and this comes at a small (and usually negligible) cost in additional computational time. Table 1 shows that the ‘extension’-method has the advantage of levelling out the performance of quasi-analytic methods across maturities, shrinking the range of MAPEs. For some methods (BAW, CS2 and CS3) the ‘extension’-method achieves remarkable reductions in MAPE of over 80%. Surprisingly, the ‘extension’-method also works efficiently for options with maturities below 6 months. For each maturity range, the percentage of ‘best solutions’ for the extended versions is above 99%. The only exception is for the integral method with 3 exercise-dates (K3), for very short-maturity options (shorter than 6 months) since in this case the percentage is only 38.1%. Furthermore, as indicated in Table 2, the ‘extension’-method sensibly solves particular problems\textsuperscript{22} encountered by standard methods and consequently the maximum relative error is much lower with the ‘extensions’ rather than with their standard counterparts.

\textsuperscript{21}For the asymptotic methods in Section 4.2, since there is no ‘standard’ version that works for maturities as long as 5 years, the number of best solutions found indicates the number of scenarios for which the relative error is the smallest among the 5 asymptotic ‘extended’ methods.

\textsuperscript{22}These problems usually occur under low-volatility regimes. Among others, Plots 7 and 8 in Broadie and Detemple (1996, pp. 1229) show that the performance of several pricing methods degrades for low-volatility regimes.
All the results relative to the extended versions of the methods are in relation to the ratio $t_x/T$ which has the lowest MAPE linked to it.\footnote{The ratio for each method is the one that minimizes the corresponding solid line in Figure 2.} However, we point out that the ‘extension’-method is robust to the selection of this ratio in the sense that for wide ranges of this ratio each ‘extended’ version outperforms the corresponding standard version. Figure 2 plots the MAPE (cumulative for the scenarios in A-J) of each ‘standard’ method against the one of the corresponding ‘extended’ method. For any ratio $t_x/T$ below 0.7, each ‘extended’ version produces MAPEs that are sensibly smaller than the ‘standard’ version. For ratios above 0.7, the ‘extension’-method converges to the method in Bjerksund and Stensland (1993) while for small ratios, it converges to the corresponding ‘standard’ method, as we were theoretically expecting. For the improved quadratic method in JZ, the range of the ratio is slightly smaller since the ‘standard’ JZ performs better than the other methods.

Table 3 summarizes the results on the three error measures discussed above for the asymptotic expansion methods of the OEP. Overall, the ‘extension’-method can be successfully used to ‘extend’ asymptotic expansions of the OEP up to maturities as long as 5 years. The method that outperforms the other ones under almost any scenarios and measures is CZ-P-m. The performances reached by the methods in Table 3 are better than some of the considered (‘standard’) quasi-analytic methods but slightly worse than their ‘extended’ versions. Also for the asymptotic expansion methods, the results are robust to the selection of the parameter $t_x$.\footnote{The results for different selection of $t_x$ are not shown in the paper and are provided by the authors upon request.}
Figure 2: Comparison of quasi-analytic ‘standard’ methods and their ‘extended’ versions: MAPE as a function of the ratio $t_x/T$.

This figure shows the ranges of ratios $t_x/T$ for which the ‘extended’ version (solid lines) outperforms the ‘standard’ version (dash-dot lines) of each method. The methods considered are the ones in Section 4.1 and the results are shown for all maturities ($\leq 5$ years). The minima of the solid lines correspond to the values shown in Table 1 in column A-J.
5.2. Hedging performance

The numerical study on hedging performance is based on the implementation of delta-hedging strategies and an analysis of the hedging errors. Methods in both Section 4.1 and Section 4.2 are considered. The performance evaluation is carried out according to the average quadratic hedging error where the hedging error is defined as the difference in value between the option and the hedging portfolio at the exercise date. The fair price for the options is chosen to be the 15,000 time-step binomial-tree price.

We consider three sets of 1000 simulated scenarios, each corresponding to an American put option with certain parameters and for each of them we set-up the delta-hedging strategy deriving from the selected method. The options have strike price $K = 100$ and maturity 5 years, are written on an underlying asset with volatilities $\sigma = 0.4$, dividend rate $\delta = 0.04$, initial spot price $S_0 = 100$ and risk-free $r = 0.05$. Three alternative values for the expected log-return on the underlying asset are considered, namely $\mu = \{0.05, 0.06, 0.07\}$ and they correspond to the three simulated sets, as shown in Table 4. The results in this table demonstrate that our proposed approach is capable of delivering an improved hedging performance when compared to the 'standard' methodologies. This conclusion is robust to changes in model parameters, choice of 'standard' method which is extended, as well as choice of option maturity.

6. Conclusion

Most of the quasi-analytic methods currently used for pricing and hedging American options are more likely to perform better for short-maturity options than for long-maturity ones. We proposed here a quasi-analytic method
which has the potential to improve the performances of any quasi-analytic pricing and hedging method for long-maturity options. We carried a numerical study on the usefulness of the ‘extension’-method for pricing and hedging a large sample of American options when combined with 6 well-known quasi-analytic methodologies. We showed that for each of them remarkable improvements were obtained for both hedging and pricing with the only loss of very little computational time, usually negligible. Remarkably, the ‘extension’-method improves also on the pricing and hedging performances of short maturity options.

Furthermore, we showed that many asymptotic expansions of the OEP near maturity can be effectively used together with analytic methods. A numerical study on 5 asymptotic expansions shows that, if they are coupled to the ‘extension’-method, good performances are reached even for options with time-to-maturity as long as 5 years.

References


**Appendices**

**A. Proof Proposition 3.1**

Given the optimal exercise price in (3) for the generic $X \leq S_{f_x}(t_x)$, let us define the correspondent stopping time\(^{25}\) as:

$$ t^* = \inf \left\{ \inf_{t \in [t_0, \infty)} \{ S_t \leq S^{(E)}_f(t) \}, T \right\} = \inf \{ t^*_0(X), t^*_x(S_{f_x}(t)), T \} $$

\(^{25}\)The intervals are right-open to avoid situations where the infimum does not exists.
where $t^*_u(x) = \inf_{t \in [t_u, \infty)} \{S_t \leq x\}$. Consequently, the American put price is:

$$P_{t_0}^{(E)}(S_{t_0}, T, K|t_x) = E_{t_0} \left[ e^{-r(t^*_u-t_0)}(K - S_{t^*_u})^+ \right] = e^{r t_0} E_{t_0} \left[ e^{-r t^*_u} (K - S_{t^*_u})^+ \right] =$$

$$= e^{r t_0} \left\{ E_{t_0} \left[ e^{-r t^*_u} (K - X) I(t_0 \leq t^*_u < t_x) \right] +
+ E_{t_0} \left[ e^{-r t^*_u} (K - S_{t^*_u}) I(t^*_u = t_x) \right] +
+ E_{t_0} \left[ e^{-r t^*_u} (K - S_f^{(E)}(t^*))^+ I(t_x < t^*_u \leq T) \right] \right\}.$$ (A.1)

The following results are helpful in the solution of (A.1). The first result is the price of a perpetual put option starting at time $u$:

$$E_u \left[ e^{-r(t^*_u-u)}(K - x) \right] = \alpha(x) S^\beta_u,$$ (A.2)

where $\alpha(x)$ and $\beta$ are given in Proposition 3.1. The second result is the following expectation:

$$\varphi_{t_0}^P(S_{t_0}, t_x|\gamma, H, X) = E_{t_0} \left[ e^{-r t_x} S^\gamma_{t_x} I(S_{t_x} > H) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] =$$

$$= X^\gamma E_{t_0} \left[ e^{-r t_x} e^{z_{t_x}} I(z_{t_x} > B_H) I \left( \inf_{t \in [t_0, t_x]} z_t > 0 \right) \right] =$$

$$= X^\gamma \int_{B_H}^{\infty} e^{-r t_x} e^z f_0(z, \gamma) dz =$$

$$= e^{\lambda t_x} S^\gamma_{t_0} \left[ N(d_{\varphi,1}(H)) - \left( \frac{X}{S_{t_0}} \right)^\kappa N(d_{\varphi,2}(H)) \right]$$ (A.3)

\footnote{We prove here the result for $\gamma > 0$. It is straightforward to prove that formula (A.3) also holds for $\gamma = 0$.}
where \( B_H = \gamma \ln \frac{H}{X}, \) \( z_t = \gamma \ln \frac{S_t}{X}, \) \( N(\cdot) \) is the standard normal cumulative distribution function (cdf),

\[
d_{\varphi,1}(H) = \frac{\ln \frac{S_{t_0}}{H} + (b + (\gamma - \frac{1}{2})\sigma^2)(t_x - t_0)}{\sigma \sqrt{t_x - t_0}},
\]

\[
d_{\varphi,2}(H) = \frac{\ln \frac{X^2}{S_{t_0}H} + (b + (\gamma - \frac{1}{2})\sigma^2)(t_x - t_0)}{\sigma \sqrt{t_x - t_0}},
\]

\( \lambda = -r + \gamma b + \frac{1}{2} \gamma (\gamma - 1)\sigma^2 \) and \( \kappa = \frac{2b}{\sigma^2} + (2\gamma - 1). \) In the above derivation we also used the expression for \( f_0(z, \gamma), \) the probability density function of an arithmetic Brownian motion at time \( t_x \) with positive initial value \( z_{t_0}, \) drift parameter \( b_2 = \gamma b_1, \) volatility parameter \( \sigma_1 = \gamma \sigma \) and an absorbing barrier at 0:

\[
f_0(z, \gamma) = \frac{n \left( z - z_{t_0} - b_2(t_x - t_0) \right) - e^{-\frac{2b_2 z_{t_0}}{\sigma_1^2}} n \left( z + z_{t_0} - b_2(t_x - t_0) \right)}{\sigma_1 \sqrt{t_x - t_0}}.
\]

(A.4)

In the following and in formula (4), we shall use

\[
f_0(z) = f_0(z, 1).
\]

(A.5)

Given these two results, we can calculate the three expectations in (A.1). The calculations are similar to [Bjerksund and Stensland (2002)]. The first
The expectation in equation (A.1) is

\[
E_{t_0} \left[ e^{-rt^*} (K - X) I (t_0 \leq t^* < t_x) \right] = E_{t_0} \left[ e^{-rt_0^*(X)} (K - X) I \left( \inf_{t \in [t_0, t_x]} S_t < X \right) \right] =
\]

\[
= E_{t_0} \left[ e^{-rt_0^*(X)} (K - X) \left( 1 - I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right) \right] \quad (\text{e.1}) =
\]

\[
= \alpha(X) S_{t_0} \beta e^{-rt_0} - \alpha(X) E_{t_0} \left[ e^{-rt_x} E_{t_x} \left[ e^{-r(t_x^*(X) - t_x)} (K - X) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] \right] \quad (\text{A.6}) =
\]

\[
= \alpha(X) S_{t_0} e^{-rt_0} - \alpha(X) \varphi_{t_0}^P (S_{t_0}, t_x | \beta, X, X)
\]
where the equivalence (e.1) follows from the definition of the stopping time and the indicator function. The second expectation in equation (A.1) is

\[
E_{t_0} \left[ e^{-r^*} (K - S_{t_x}) I (t^* = t_x) \right] = \\
= E_{t_0} \left[ e^{-r t_x} (K - S_{t_x}) I (X \leq S_{t_x} \leq S_f^{(E)}(t_x)) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] = \\
= E_{t_0} \left[ e^{-r t_x} (K - S_{t_x}) \left[ I (S_{t_x} \geq X) - I \left( S_{t_x} \geq S_f^{(E)}(t_x) \right) \right] I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] = \\
= KE_{t_0} \left[ e^{-r t_x} I (S_{t_x} \geq X) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] + \\
- KE_{t_0} \left[ e^{-r t_x} I (S_{t_x} \geq S_f^{(E)}(t_x)) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] + \\
- E_{t_0} \left[ e^{-r t_x} S_{t_x} I (S_{t_x} \geq X) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] + \\
+ E_{t_0} \left[ e^{-r t_x} S_{t_x} I \left( S_{t_x} \geq S_f^{(E)}(t_x) \right) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] = \\
= K \left[ \varphi_{t_0}^P(S_{t_0}, t_x|0, X, X) - \varphi_{t_0}^P(S_{t_0}, t_x|0, S_f^{(E)}(t_x), X) \right] + \\
- \left[ \varphi_{t_0}^P(S_{t_0}, t_x|1, X, X) - \varphi_{t_0}^P(S_{t_0}, t_x|1, S_f^{(E)}(t_x), X) \right]. \quad (A.7)
\]

Finally, the third expectation in equation (A.1) is

\[
E_{t_0} \left[ e^{-r^*} (K - S_f^{(E)}(t^*))^+ I (t_x < t^* \leq T) \right] = \\
= e^{-r t_x} E_{t_x} \left[ E_{t_x} \left[ e^{-r(t^*(\gamma_f^{(E)}(t^*))-t_x)} (K - S_f^{(E)}(t^*))^+ \right] I (t_x < t^* \leq T) \right] \quad \text{(e.2)} \]

\[
= e^{-r t_x} E_{t_0} \left[ P_{t_x}(S_{t_x}, T, K) I (S_{t_x} > S_f^{(E)}(t_x)) I \left( \inf_{t \in [t_0, t_x]} S_t > X \right) \right] = \\
= e^{-r t_x} E_{t_0} \left[ P_{t_x}(X e^{-\alpha t_x}, T, K) I (z_{t_x} > B) I \left( \inf_{t \in [t_0, t_x]} z_t > 0 \right) \right] = \\
= e^{-r t_x} \int^{+\infty}_B P(X e^{-z}, t_x, T, K) f_0(z) dz \quad (A.8)
\]
where the equivalence (e.2) follows since the inner expectation is the time-\(t_x\) price of an option with maturity \(T\) and strike price \(K\), i.e. the short-maturity option. In (A.8), \(z_{t_x} = \gamma \ln \frac{S_{t_x}}{X}\), for \(\gamma = 1\). By summing up equations (A.6), (A.7) and (A.8) as in (A.1), we have equation (4).

B. Proof Proposition 3.2

The proof consists in showing that

\[
\lim_{T \to +\infty} P_{t_0}^{(E)}(S_{t_0}, T, K|t_x, X) = \alpha(X)S_{t_0}^\beta,
\]

(B.1)

for any selection of \(t_x \in (t_0, T]\) and \(X\). We indicate \(t_x = \vartheta T\) with \(\vartheta \in (0, 1]\) and prove that the result is independent of \(\vartheta\). We discuss first the case for \(\gamma = \beta\) and then the cases for \(\gamma = 0\) and \(\gamma = 1\).

For \(\gamma = \beta\), \(\gamma < \frac{\sigma^2}{2} - b\) holds and consequently

\[
\lim_{T \to +\infty} d_{\varphi,1}(H) = \lim_{T \to +\infty} d_{\varphi,2}(H) = -\infty
\]

and

\[
\lim_{T \to +\infty} \varphi_{t_0}^P(S_{t_0}, t_x|\beta, X, X) = 0.
\]

(B.2)

Note that (B.2) also holds when \(\lambda > 0\), since l’Hôpital’s rule guarantees that

\[
\lim_{T \to +\infty} e^{\lambda(t_x-t_0)} N(d_{\varphi,1}(H)) = \lim_{T \to +\infty} e^{\lambda(t_x-t_0)} N(d_{\varphi,2}(H)) = 0.
\]

For \(\gamma = 0\) or \(\gamma = 1\), since \(\lambda = -r+\gamma b\) is non-positive and \(\lim_{T \to +\infty} d_{\varphi,1}(H) = \lim_{T \to +\infty} d_{\varphi,2}(H) = v\), with \(v\) independent from \(H\). Then the limit

\[
\lim_{T \to +\infty} \varphi_{t_0}^P(S_{t_0}, t_x|\gamma, H, X)
\]

(B.3)

39
is finite for any positive finite $H$ and therefore:

\[
\lim_{T \to +\infty} \left[ \varphi^P_{t_0}(S_{t_0}, t_x \mid \gamma, H_1, X) - \varphi^P_{t_0}(S_{t_0}, t_x \mid \gamma, H_2, X) \right] = 0 \quad (B.4)
\]

for any finite $H_1$ and $H_2$. Finally,

\[
\lim_{T \to +\infty} \int_{B}^{+\infty} g(z) \, dz = \lim_{T \to +\infty} \int_{B}^{+\infty} e^{-r(t_x-t_0)} P_{t_x}(X e^{-z}, T, K) f_0(z) \, dz = 0 \quad (B.5)
\]

since \(\lim_{T \to +\infty} f_0(z) = 0\), \(\lim_{T \to +\infty} e^{-r(t_x-t_0)} = \lim_{T \to +\infty} e^{-r(\partial T - t_0)} = 0\), and \((0 \leq P_{t_x}(X e^{-z}, T, K) \leq K)\). Consequently, the limit in \(\text{(B.1)}\) and the Proposition \(3.2\) follow since the quantities $\alpha(X)$ and $\beta$ are time invariant.

C. Expressions for terms in Proposition \[3.3\]

Defining $n(\cdot)$ as the standard normal density function, these two terms are useful for Proposition \[3.3\]

\[
f'_0(z) = \frac{1}{S_{t_0} \sigma \sqrt{t_x - t_0}} \times \left[ e^{-\frac{2 b_1 z_{t_0}}{\sigma^2}} n \left( \frac{z + z_{t_0} - b_1(t_x - t_0)}{\sigma \sqrt{t_x - t_0}} \right) \left( \frac{z + z_{t_0} - b_1(t_x - t_0)}{\sigma^2(t_x - t_0)} + \frac{2b_1}{\sigma^2} \right) \right.
\]
\[
+ \frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma^2(t_x - t_0)} \left( \frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma \sqrt{t_x - t_0}} \right) \left( \frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma^2(t_x - t_0)} \right) \]\n
\]

40
and

\[
\varphi'_{t_0,S_0}(S_{t_0}, t_x|\gamma, H, X) = \frac{\partial \varphi_{t_0}^P(S_{t_0}, t_x|\gamma, H, X)}{\partial S_{t_0}} = e^{\lambda t_x} [\gamma S_{t_0}^{\gamma-1} N(d_{\varphi,1}(H)) \\
+ \frac{S_{t_0}^{\gamma-1}}{\sigma \sqrt{t_x - t_0}} n(d_{\varphi,1}(H)) - X^\kappa (\gamma - \kappa) S_{t_0}^{\gamma-\kappa-1} N(d_{\varphi,2}(H)) \\
+ \frac{X^\kappa}{S_{t_0}^{\kappa-\gamma+1} \sigma \sqrt{t_x - t_0}} n(d_{\varphi,2}(H))].
\]
Table 1: Comparison of quasi-analytic ‘standard’ methods and their ‘extended’ versions: MAPE.

This table presents the mean absolute percentage error (MAPE) for 6 quasi-analytic methods (indicated as ‘S’ for standard) and their ‘extended’ version (indicated as ‘E’), i.e., when the ‘extension’-method in Proposition 3.1 is applied to them. The methods considered are: (GJ) the method in Geske and Johnson (1984) with two and three exercise dates, (BAW) the quadratic method in Barone-Adesi and Whaley (1987), (LI) the interpolation method in Li (2010b), (K) the ‘integral’ method in Kim (1990) with two and three exercise dates, (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates, (JZ) the improved quadratic method in Ju and Zhong (1999). Ten ranges of maturity (in years) are considered: (0; 0.5] (A), (0.5; 1] (B), (1; 1.5] (C), (1.5; 2] (D), (2; 2.5] (E), (2.5; 3] (F), (3; 3.5] (G), (3.5; 4] (H), (4; 4.5] (I), (4.5; 5] (J). The results are based on 1000 simulated scenarios for each maturity range drawn from the distribution indicated in Broadie and Detemple (1996) and summarized in Section 5. In the last row, for each maturity range, there is the number of option with fair price above 0.5. The values for the ‘extension’-method are calculated for the ratio $t_x/T$ which has the lowest MAPE linked to it (the ratio for each method is the one corresponding to the minimum of the solid lines in Figure 2).

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>A-J</th>
</tr>
</thead>
<tbody>
<tr>
<td>GJ2</td>
<td>S</td>
<td>0.389%</td>
<td>0.871%</td>
<td>1.343%</td>
<td>1.893%</td>
<td>2.084%</td>
<td>2.622%</td>
<td>3.131%</td>
<td>3.587%</td>
<td>3.984%</td>
<td>4.350%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.244%</td>
<td>0.433%</td>
<td>0.552%</td>
<td>0.653%</td>
<td>0.682%</td>
<td>0.751%</td>
<td>0.816%</td>
<td>0.823%</td>
<td>0.845%</td>
<td>0.843%</td>
</tr>
<tr>
<td>GJ3</td>
<td>S</td>
<td>0.276%</td>
<td>0.608%</td>
<td>0.930%</td>
<td>1.295%</td>
<td>1.417%</td>
<td>1.788%</td>
<td>2.112%</td>
<td>2.376%</td>
<td>2.648%</td>
<td>2.902%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.195%</td>
<td>0.328%</td>
<td>0.436%</td>
<td>0.511%</td>
<td>0.523%</td>
<td>0.573%</td>
<td>0.617%</td>
<td>0.623%</td>
<td>0.650%</td>
<td>0.668%</td>
</tr>
<tr>
<td>BAW</td>
<td>S</td>
<td>0.155%</td>
<td>0.395%</td>
<td>0.632%</td>
<td>0.900%</td>
<td>1.202%</td>
<td>1.429%</td>
<td>1.696%</td>
<td>1.945%</td>
<td>2.252%</td>
<td>2.380%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.119%</td>
<td>0.140%</td>
<td>0.192%</td>
<td>0.237%</td>
<td>0.239%</td>
<td>0.268%</td>
<td>0.288%</td>
<td>0.298%</td>
<td>0.337%</td>
<td>0.337%</td>
</tr>
<tr>
<td>LI</td>
<td>S</td>
<td>0.169%</td>
<td>0.293%</td>
<td>0.515%</td>
<td>0.669%</td>
<td>0.879%</td>
<td>1.053%</td>
<td>1.288%</td>
<td>1.532%</td>
<td>1.821%</td>
<td>1.965%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.087%</td>
<td>0.131%</td>
<td>0.205%</td>
<td>0.247%</td>
<td>0.256%</td>
<td>0.320%</td>
<td>0.359%</td>
<td>0.393%</td>
<td>0.395%</td>
<td>0.410%</td>
</tr>
<tr>
<td>K2</td>
<td>S</td>
<td>0.248%</td>
<td>0.404%</td>
<td>0.578%</td>
<td>0.713%</td>
<td>0.794%</td>
<td>0.899%</td>
<td>0.988%</td>
<td>1.182%</td>
<td>1.291%</td>
<td>1.488%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.180%</td>
<td>0.271%</td>
<td>0.348%</td>
<td>0.404%</td>
<td>0.450%</td>
<td>0.476%</td>
<td>0.500%</td>
<td>0.503%</td>
<td>0.535%</td>
<td>0.534%</td>
</tr>
<tr>
<td>K3</td>
<td>S</td>
<td>0.125%</td>
<td>0.198%</td>
<td>0.275%</td>
<td>0.346%</td>
<td>0.358%</td>
<td>0.445%</td>
<td>0.518%</td>
<td>0.649%</td>
<td>0.678%</td>
<td>0.862%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.128%</td>
<td>0.129%</td>
<td>0.158%</td>
<td>0.177%</td>
<td>0.189%</td>
<td>0.203%</td>
<td>0.217%</td>
<td>0.229%</td>
<td>0.249%</td>
<td>0.256%</td>
</tr>
<tr>
<td>CS2</td>
<td>S</td>
<td>0.102%</td>
<td>0.146%</td>
<td>0.155%</td>
<td>0.189%</td>
<td>0.182%</td>
<td>0.281%</td>
<td>0.400%</td>
<td>0.566%</td>
<td>0.688%</td>
<td>0.867%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.063%</td>
<td>0.084%</td>
<td>0.079%</td>
<td>0.069%</td>
<td>0.058%</td>
<td>0.051%</td>
<td>0.057%</td>
<td>0.069%</td>
<td>0.069%</td>
<td>0.069%</td>
</tr>
<tr>
<td>CS3</td>
<td>S</td>
<td>0.067%</td>
<td>0.086%</td>
<td>0.091%</td>
<td>0.115%</td>
<td>0.112%</td>
<td>0.173%</td>
<td>0.243%</td>
<td>0.341%</td>
<td>0.412%</td>
<td>0.517%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.040%</td>
<td>0.049%</td>
<td>0.046%</td>
<td>0.040%</td>
<td>0.035%</td>
<td>0.036%</td>
<td>0.038%</td>
<td>0.044%</td>
<td>0.056%</td>
<td>0.062%</td>
</tr>
<tr>
<td>JZ</td>
<td>S</td>
<td>0.051%</td>
<td>0.108%</td>
<td>0.141%</td>
<td>0.167%</td>
<td>0.180%</td>
<td>0.205%</td>
<td>0.234%</td>
<td>0.260%</td>
<td>0.290%</td>
<td>0.305%</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>0.044%</td>
<td>0.076%</td>
<td>0.096%</td>
<td>0.105%</td>
<td>0.105%</td>
<td>0.106%</td>
<td>0.112%</td>
<td>0.108%</td>
<td>0.108%</td>
<td>0.108%</td>
</tr>
</tbody>
</table>
Table 2: Comparison of quasi-analytic ‘standard’ methods and their ‘extended’ versions: maximum relative error.

This table presents for 6 quasi-analytic methods (indicated as ‘S’ for standard) and their ‘extended’ version (indicated as ‘E’) the maximum relative error. The methods considered are: (GJ) the method in Geske and Johnson (1984) with two and three exercise dates, (BAW) the quadratic method in Barone-Adesi and Whaley (1987), (LI) the interpolation method in Li (2010b), (K) the ‘integral’ method in Kim (1990) with two and three exercise dates, (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates, (JZ) the improved quadratic method in Ju and Zhong (1999). Ten ranges of maturity (in years) are considered: (0; 0.5] (A), (0.5; 1] (B), (1; 1.5] (C), (1.5; 2] (D), (2; 2.5] (E), (2.5; 3] (F), (3; 3.5] (G), (3.5; 4] (H), (4; 4.5] (I), (4.5; 5] (J). The results are based on 1000 simulated scenarios for each maturity range drawn from the distribution indicated in Broadie and Detemple (1996) and summarized in Section 5.

<table>
<thead>
<tr>
<th>Method</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>A-J</th>
</tr>
</thead>
<tbody>
<tr>
<td>GJ2</td>
<td>S 4.866%</td>
<td>10.754%</td>
<td>24.208%</td>
<td>32.925%</td>
<td>17.259%</td>
<td>29.542%</td>
<td>27.883%</td>
<td>35.698%</td>
<td>35.491%</td>
<td>34.591%</td>
<td>43.737%</td>
</tr>
<tr>
<td></td>
<td>E 1.971%</td>
<td>3.458%</td>
<td>5.222%</td>
<td>4.838%</td>
<td>5.176%</td>
<td>5.221%</td>
<td>5.898%</td>
<td>3.838%</td>
<td>3.832%</td>
<td>4.052%</td>
<td>5.898%</td>
</tr>
<tr>
<td>GJ3</td>
<td>S 3.106%</td>
<td>6.751%</td>
<td>15.898%</td>
<td>22.218%</td>
<td>12.231%</td>
<td>19.700%</td>
<td>18.470%</td>
<td>30.427%</td>
<td>30.663%</td>
<td>23.497%</td>
<td>30.663%</td>
</tr>
<tr>
<td></td>
<td>E 1.425%</td>
<td>3.204%</td>
<td>5.434%</td>
<td>3.464%</td>
<td>3.680%</td>
<td>3.711%</td>
<td>4.181%</td>
<td>4.208%</td>
<td>4.315%</td>
<td>4.272%</td>
<td>5.434%</td>
</tr>
<tr>
<td>BAW</td>
<td>S 2.866%</td>
<td>7.260%</td>
<td>5.773%</td>
<td>11.854%</td>
<td>11.289%</td>
<td>12.121%</td>
<td>15.509%</td>
<td>14.205%</td>
<td>13.830%</td>
<td>14.592%</td>
<td>15.509%</td>
</tr>
<tr>
<td></td>
<td>E 1.317%</td>
<td>2.133%</td>
<td>1.556%</td>
<td>3.748%</td>
<td>2.258%</td>
<td>2.562%</td>
<td>3.201%</td>
<td>3.117%</td>
<td>2.887%</td>
<td>2.849%</td>
<td>3.748%</td>
</tr>
<tr>
<td>LI</td>
<td>S 1.669%</td>
<td>2.747%</td>
<td>9.944%</td>
<td>8.912%</td>
<td>34.387%</td>
<td>7.363%</td>
<td>8.746%</td>
<td>11.622%</td>
<td>11.578%</td>
<td>14.179%</td>
<td>34.387%</td>
</tr>
<tr>
<td></td>
<td>E 0.752%</td>
<td>1.241%</td>
<td>4.055%</td>
<td>3.053%</td>
<td>2.542%</td>
<td>2.435%</td>
<td>3.122%</td>
<td>3.265%</td>
<td>2.831%</td>
<td>2.816%</td>
<td>4.055%</td>
</tr>
<tr>
<td></td>
<td>E 1.319%</td>
<td>1.385%</td>
<td>1.275%</td>
<td>1.510%</td>
<td>1.990%</td>
<td>2.082%</td>
<td>1.707%</td>
<td>1.847%</td>
<td>1.816%</td>
<td>1.515%</td>
<td>2.082%</td>
</tr>
<tr>
<td>K3</td>
<td>S 1.670%</td>
<td>2.384%</td>
<td>10.140%</td>
<td>10.705%</td>
<td>4.135%</td>
<td>10.923%</td>
<td>8.444%</td>
<td>16.484%</td>
<td>15.517%</td>
<td>18.420%</td>
<td>18.420%</td>
</tr>
<tr>
<td></td>
<td>E 1.315%</td>
<td>0.745%</td>
<td>3.234%</td>
<td>2.095%</td>
<td>1.262%</td>
<td>2.120%</td>
<td>1.485%</td>
<td>2.778%</td>
<td>2.773%</td>
<td>1.858%</td>
<td>3.234%</td>
</tr>
<tr>
<td>CS2</td>
<td>S 0.852%</td>
<td>1.245%</td>
<td>6.315%</td>
<td>9.556%</td>
<td>3.786%</td>
<td>7.127%</td>
<td>8.374%</td>
<td>18.115%</td>
<td>12.442%</td>
<td>15.902%</td>
<td>18.115%</td>
</tr>
<tr>
<td></td>
<td>E 0.561%</td>
<td>0.701%</td>
<td>2.093%</td>
<td>1.238%</td>
<td>0.538%</td>
<td>0.872%</td>
<td>1.007%</td>
<td>2.237%</td>
<td>1.209%</td>
<td>0.701%</td>
<td>2.237%</td>
</tr>
<tr>
<td>CS3</td>
<td>S 0.824%</td>
<td>0.874%</td>
<td>3.511%</td>
<td>6.153%</td>
<td>2.644%</td>
<td>4.477%</td>
<td>5.895%</td>
<td>11.622%</td>
<td>6.820%</td>
<td>9.524%</td>
<td>11.622%</td>
</tr>
<tr>
<td></td>
<td>E 0.428%</td>
<td>0.442%</td>
<td>1.244%</td>
<td>0.733%</td>
<td>0.453%</td>
<td>0.531%</td>
<td>0.601%</td>
<td>1.328%</td>
<td>1.342%</td>
<td>0.827%</td>
<td>1.342%</td>
</tr>
<tr>
<td>JZ</td>
<td>S 0.438%</td>
<td>1.521%</td>
<td>2.682%</td>
<td>2.856%</td>
<td>2.912%</td>
<td>3.028%</td>
<td>3.863%</td>
<td>3.815%</td>
<td>3.624%</td>
<td>3.701%</td>
<td>3.863%</td>
</tr>
<tr>
<td></td>
<td>E 0.260%</td>
<td>0.665%</td>
<td>1.459%</td>
<td>1.359%</td>
<td>1.053%</td>
<td>1.049%</td>
<td>1.455%</td>
<td>1.440%</td>
<td>1.513%</td>
<td>1.410%</td>
<td>1.513%</td>
</tr>
</tbody>
</table>
Table 3: Pricing performance of ‘extended’ asymptotic expansion of the OEP near maturity.

This table presents for 5 asymptotic expansions methods (coupled with the ‘extension’-method) the mean absolute percentage error (MAPE), the maximum relative error and the percentage of ‘best’ solutions. The methods considered are: (EKK) the method in Evans et al. (2002), (ZL) the method in Zhang and Li (2010), (CZ),(CZ-P)-(CZ-P-m) the method in Cheng and Zhang (2012) basic, with Padé’ approximation and with Padé’ approximation corrected for Evans et al. (2002), respectively.

Ten ranges of maturities (in years) are considered: (0; 0.5] (A), (0.5; 1] (B), (1; 1.5] (C), (1.5; 2] (D), (2; 2.5] (E), (2.5; 3] (F), (3;3.5] (G), (3.5;4] (H), (4;4.5] (I), (4.5;5] (J). The results are based on 1000 simulated scenarios for each maturity range drawn from the distribution indicated in Broadie and Detemple (1996) and summarized in Section 5. In the last row, for each maturity range, there is the number of option with fair price above 0.5 and maturities longer than 2 weeks. The values for the ‘extension’-method are calculated for a value \( T - t_x \) (in weeks) which has the smallest MAPE linked to it. All the results are calculated considering one exercise date per day (7 days a week).

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>A-J</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MAPE</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EKK</td>
<td>0.143%</td>
<td>0.322%</td>
<td>0.487%</td>
<td>0.732%</td>
<td>0.568%</td>
<td>0.502%</td>
<td>0.505%</td>
<td>0.703%</td>
<td>0.897%</td>
<td>1.172%</td>
<td>0.617%</td>
</tr>
<tr>
<td>ZL</td>
<td>0.208%</td>
<td>0.496%</td>
<td>0.656%</td>
<td>0.788%</td>
<td>0.686%</td>
<td>0.660%</td>
<td>0.741%</td>
<td>0.942%</td>
<td>1.209%</td>
<td>1.446%</td>
<td>0.800%</td>
</tr>
<tr>
<td>CZ</td>
<td>0.323%</td>
<td>0.499%</td>
<td>0.611%</td>
<td>0.954%</td>
<td>0.796%</td>
<td>0.679%</td>
<td>0.720%</td>
<td>0.925%</td>
<td>1.057%</td>
<td>1.448%</td>
<td>0.816%</td>
</tr>
<tr>
<td>CZ-P</td>
<td>0.347%</td>
<td>0.604%</td>
<td>0.800%</td>
<td>1.188%</td>
<td>0.943%</td>
<td>0.831%</td>
<td>0.807%</td>
<td>1.042%</td>
<td>1.256%</td>
<td>1.527%</td>
<td>0.952%</td>
</tr>
<tr>
<td>CZ-P-m</td>
<td>0.121%</td>
<td>0.181%</td>
<td>0.242%</td>
<td>0.480%</td>
<td>0.371%</td>
<td>0.229%</td>
<td>0.261%</td>
<td>0.468%</td>
<td>0.759%</td>
<td>1.037%</td>
<td>0.424%</td>
</tr>
<tr>
<td><strong>Max Error</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EKK</td>
<td>3.123%</td>
<td>4.612%</td>
<td>5.096%</td>
<td>6.717%</td>
<td>5.801%</td>
<td>5.293%</td>
<td>4.354%</td>
<td>4.221%</td>
<td>8.214%</td>
<td>8.184%</td>
<td>8.214%</td>
</tr>
<tr>
<td>ZL</td>
<td>4.546%</td>
<td>6.624%</td>
<td>8.681%</td>
<td>9.104%</td>
<td>7.988%</td>
<td>6.936%</td>
<td>5.943%</td>
<td>5.265%</td>
<td>5.414%</td>
<td>6.393%</td>
<td>9.104%</td>
</tr>
<tr>
<td>CZ</td>
<td>3.528%</td>
<td>4.256%</td>
<td>4.566%</td>
<td>5.415%</td>
<td>5.837%</td>
<td>5.166%</td>
<td>4.663%</td>
<td>5.061%</td>
<td>4.785%</td>
<td>5.323%</td>
<td>5.837%</td>
</tr>
<tr>
<td>CZ-P</td>
<td>3.577%</td>
<td>4.898%</td>
<td>5.290%</td>
<td>6.960%</td>
<td>6.516%</td>
<td>6.320%</td>
<td>5.357%</td>
<td>5.917%</td>
<td>6.110%</td>
<td>5.461%</td>
<td>6.960%</td>
</tr>
<tr>
<td>CZ-P-m</td>
<td>4.242%</td>
<td>4.150%</td>
<td>4.776%</td>
<td>2.543%</td>
<td>2.269%</td>
<td>2.103%</td>
<td>2.871%</td>
<td>2.920%</td>
<td>4.054%</td>
<td>5.461%</td>
<td>5.461%</td>
</tr>
<tr>
<td><strong>% Best</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EKK</td>
<td>70.450%</td>
<td>54.129%</td>
<td>42.530%</td>
<td>40.618%</td>
<td>39.577%</td>
<td>34.036%</td>
<td>32.803%</td>
<td>35.406%</td>
<td>38.771%</td>
<td>35.559%</td>
<td>41.913%</td>
</tr>
<tr>
<td>ZL</td>
<td>69.675%</td>
<td>44.346%</td>
<td>45.085%</td>
<td>53.185%</td>
<td>43.203%</td>
<td>37.251%</td>
<td>28.122%</td>
<td>26.415%</td>
<td>22.513%</td>
<td>21.353%</td>
<td>38.254%</td>
</tr>
<tr>
<td>CZ</td>
<td>32.800%</td>
<td>40.289%</td>
<td>37.323%</td>
<td>28.235%</td>
<td>33.050%</td>
<td>41.270%</td>
<td>41.711%</td>
<td>45.127%</td>
<td>51.538%</td>
<td>50.683%</td>
<td>40.424%</td>
</tr>
<tr>
<td>CZ-P</td>
<td>8.000%</td>
<td>18.228%</td>
<td>3.043%</td>
<td>4.511%</td>
<td>4.055%</td>
<td>10.370%</td>
<td>16.437%</td>
<td>10.571%</td>
<td>4.531%</td>
<td>3.477%</td>
<td>8.301%</td>
</tr>
<tr>
<td>CZ-P-m</td>
<td>51.331%</td>
<td>50.576%</td>
<td>35.660%</td>
<td>33.798%</td>
<td>32.090%</td>
<td>37.778%</td>
<td>50.898%</td>
<td>47.890%</td>
<td>35.789%</td>
<td>35.331%</td>
<td>40.829%</td>
</tr>
</tbody>
</table>
Table 4: Hedging comparison for different maturities.

This table presents the average quadratic hedging error for 6 quasi-analytic methods (indicated as ‘S’ for standard), their ‘extended’ versions (indicated as ‘E’) and 5 ‘extended’ asymptotic-expansion methods. The considered quasi-analytic methods are: (GJ) the method in Geske and Johnson (1984) with two and three exercise dates, (BAW) the quadratic method in Barone-Adesi and Whaley (1987), (LI) the interpolation method in Li (2010b), (K) the ‘integral’ method in Kim (1990) with two and three exercise dates, (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates, (JZ) the improved quadratic method in Ju and Zhong (1999). However, the considered asymptotic expansion methods are: (EKK) the method in Evans et al. (2002), (ZL) the method in Zhang and Li (2010), (CZ), (CZ-P) and (CZ-P-m) the method in Cheng and Zhang (2012) basic, with Pade’ approximation and with Pade’ approximation corrected for Evans et al. (2002), respectively. The results are based on 3 sets of 1000 simulated scenarios each with a different expected log-return for the underlying asset: 0.05, 0.06 and 0.07. The parameters are \( r = 0.05 \), \( \delta = 0.04 \), \( K = 100 \), \( \sigma = 0.4 \) and \( S_0 = 100 \). The analysis is based on monthly hedging rolling frequency. The results are presented for five different maturities from 1 to 5 years.

<table>
<thead>
<tr>
<th>Time-to-maturity</th>
<th>( \mu = r = 5% )</th>
<th>( \mu = 6% )</th>
<th>( \mu = 7% )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 (year) 2 3 4 5</td>
<td>1 2 3 4 5</td>
<td>1 2 3 4 5</td>
</tr>
<tr>
<td>GJ2 S</td>
<td>0.551 0.792 1.001 1.195 1.356</td>
<td>0.631 0.889 1.101 1.279 1.403</td>
<td>0.562 0.810 1.032 1.218 1.368</td>
</tr>
<tr>
<td>GJ2 E</td>
<td>0.470 0.705 0.918 1.124 1.318</td>
<td>0.468 0.664 0.858 1.069 1.214</td>
<td>0.479 0.726 0.946 1.150 1.331</td>
</tr>
<tr>
<td>GJ3 S</td>
<td>0.553 0.796 1.009 1.207 1.371</td>
<td>0.633 0.895 1.111 1.293 1.419</td>
<td>0.563 0.814 1.039 1.229 1.382</td>
</tr>
<tr>
<td>GJ3 E</td>
<td>0.470 0.705 0.917 1.123 1.317</td>
<td>0.468 0.663 0.857 1.069 1.213</td>
<td>0.479 0.725 0.945 1.150 1.330</td>
</tr>
<tr>
<td>BAW S</td>
<td>0.554 0.800 1.016 1.217 1.383</td>
<td>0.639 0.907 1.127 1.312 1.438</td>
<td>0.564 0.818 1.046 1.240 1.394</td>
</tr>
<tr>
<td>BAW E</td>
<td>0.470 0.704 0.916 1.121 1.314</td>
<td>0.468 0.663 0.856 1.066 1.210</td>
<td>0.479 0.725 0.944 1.147 1.327</td>
</tr>
<tr>
<td>LI S</td>
<td>0.554 0.800 1.016 1.218 1.384</td>
<td>0.639 0.907 1.128 1.313 1.439</td>
<td>0.565 0.818 1.047 1.241 1.395</td>
</tr>
<tr>
<td>LI E</td>
<td>0.458 0.666 0.866 1.072 1.256</td>
<td>0.445 0.632 0.818 1.020 1.160</td>
<td>0.438 0.679 0.891 1.092 1.263</td>
</tr>
<tr>
<td>K2 S</td>
<td>0.551 0.786 0.986 1.169 1.320</td>
<td>0.635 0.892 1.097 1.264 1.376</td>
<td>0.561 0.804 1.017 1.192 1.332</td>
</tr>
<tr>
<td>K2 E</td>
<td>0.470 0.705 0.917 1.123 1.317</td>
<td>0.468 0.663 0.857 1.068 1.213</td>
<td>0.479 0.725 0.945 1.150 1.330</td>
</tr>
<tr>
<td>K3 S</td>
<td>0.554 0.784 0.992 1.177 1.328</td>
<td>0.639 0.897 1.103 1.272 1.384</td>
<td>0.564 0.809 1.023 1.200 1.340</td>
</tr>
<tr>
<td>K3 E</td>
<td>0.550 0.673 0.838 1.120 1.172</td>
<td>0.578 0.672 0.820 0.986 1.098</td>
<td>0.402 0.618 0.814 0.998 1.144</td>
</tr>
<tr>
<td>CS2 S</td>
<td>0.555 0.802 1.020 1.224 1.394</td>
<td>0.639 0.908 1.131 1.319 1.447</td>
<td>0.565 0.820 1.051 1.247 1.404</td>
</tr>
<tr>
<td>CS2 E</td>
<td>0.461 0.659 0.861 1.065 1.241</td>
<td>0.479 0.636 0.820 1.014 1.146</td>
<td>0.424 0.662 0.876 1.073 1.236</td>
</tr>
<tr>
<td>CS3 S</td>
<td>0.555 0.802 1.019 1.223 1.392</td>
<td>0.639 0.908 1.130 1.318 1.446</td>
<td>0.565 0.820 1.050 1.246 1.403</td>
</tr>
<tr>
<td>CS3 E</td>
<td>0.485 0.672 0.869 1.072 1.248</td>
<td>0.504 0.649 0.829 1.022 1.154</td>
<td>0.427 0.664 0.878 1.075 1.239</td>
</tr>
<tr>
<td>JZ S</td>
<td>0.554 0.801 1.018 1.221 1.388</td>
<td>0.639 0.908 1.129 1.316 1.442</td>
<td>0.565 0.819 1.048 1.243 1.398</td>
</tr>
<tr>
<td>JZ E</td>
<td>0.470 0.704 0.916 1.122 1.315</td>
<td>0.468 0.663 0.857 1.067 1.212</td>
<td>0.479 0.725 0.944 1.148 1.329</td>
</tr>
<tr>
<td>EKK</td>
<td>0.474 0.762 1.001 1.205 1.369</td>
<td>0.560 0.870 1.111 1.297 1.422</td>
<td>0.476 0.779 1.021 1.220 1.374</td>
</tr>
<tr>
<td>ZL</td>
<td>0.558 0.806 1.028 1.242 1.390</td>
<td>0.644 0.914 1.147 1.341 1.445</td>
<td>0.570 0.824 1.062 1.266 1.398</td>
</tr>
<tr>
<td>CZ</td>
<td>0.392 0.723 0.978 1.193 1.360</td>
<td>0.495 0.829 1.087 1.280 1.410</td>
<td>0.397 0.737 0.996 1.204 1.362</td>
</tr>
<tr>
<td>CZ-P &amp; CZ-P-m</td>
<td>0.433 0.745 0.992 1.201 1.362</td>
<td>0.526 0.850 1.101 1.290 1.414</td>
<td>0.432 0.757 1.010 1.212 1.364</td>
</tr>
</tbody>
</table>