Dividend Derivatives

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Abstract

Dividend derivatives are not simply a by-product of equity derivatives. They constitute a distinct growing market and an entire suite of dividend derivatives are offered to investors. In this paper we look at two potential models for equity index dividends and discuss their theoretical and practical merits. The main results emerge from a downward jump-diffusion model with beta distributed jumps and a stochastic logistic diffusion model, both able to capture the particular dynamics observed for dividends and cum-dividends, respectively, in the market. Smile calibration results are discussed with market data on Dow Jones Euro STOXX50 DVP® dividend index for futures and European call and put options.

\textit{JEL:G13, C51}

\textit{Key words:} dividend derivatives, stochastic logistic diffusion, market price of risk, smile calibration

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1 Introduction

Dividends played a major role in the development of equity financial products over the years. Lu and Karaban (2009) showed that since 1926, dividends have represented approximately one-third of total returns, the rest coming from capital appreciation. Moreover, total dividend income has increased in U.S. six-fold between 1988 and 2008, reaching almost 800 billion USD. While dividends have grown in proportion to increasing stock market capitalization, evidence shows that dividends have also grown as a proportion of personal income. Furthermore, dividends are considered a good hedge against rising inflation and they have in general lower volatility than equities as discovered by Shiller (1981).

Dividend risk is traded through many type of contracts from single-stock and index to swaps, steepeners, yield trades, ETFs, options, knock-out dividend swaps, dividend yield swap and even swaptions. Brennan (1998) suggested to strip off the equity index from its dividends and create a market in the dividend strips which should improve the informational efficiency in the economy. Another financial innovation designed to offer dividend protection is the endowment warrant, although, as discussed by Brown and Davis (2004), the protection is only partial and pricing is not easy since it is a long-term option having a stochastic strike price driven by the cash-flow of dividends.

Dividend derivatives have been traded over-the-counter (OTC) for some time, mainly in the form of index dividend swaps. The first time dividend derivatives were traded on an exchange was in 2002 in South-Africa, see Wilkens and Wimschulte (2010), but the success was moderate. NYSE Liffe have launched futures contracts on the FTSE100® dividend index in May 2009. The futures contract on the Dow Jones Euro STOXX50 DVP® index introduced on 30 June 2008 by Eurex has experienced a meteoric development. This is hardly surprising since reinvested dividends accounted for almost half of the Dow Jones Euro STOXX50® total returns since the end of December 1991.
There is a buoyant market now driven by these contracts, establishing dividends as an asset class of its own as argued by Manley and Mueller-Glissmann (2008). As of September 2014 the market in the futures on the dividends generated by the Dow Jones Euro STOXX50® has reached the level of 700000 contracts as indicated by the volumes illustrated in Figure 1 and also a growing market in dividend options as shown in Figure 2.
Figure 2: The number of traded contracts and open interest in the Options on EuroStoxx 50 Dividend Futures traded on Eurex as of September 2014. Total volume has reached 300000 contracts covering all ten maturities. Source: Eurex.

Dividend derivatives have many applications for investors. Equity derivatives traders and structured products engineers must consider their dividend risk and manage its risk. Portfolio managers with convertible bond positions and equity positions have exposure to dividend risk. In some countries investing in dividends offer a degree of tax reduction. Last but not least, carrying equity stock during systemic crises may imply less dividend payments.
than expected so by taking positions on dividend derivatives the investor on exchanges may help avoiding liquidity pressures.

As with many other emerging asset classes, the modeling for dividend derivatives lags behind in development, although modeling dividends has preoccupied academics and practitioners for many years. The literature on dividend derivatives pricing is very sparse. In general the models covering dividends modeling assume the dividend payments have either known size or timing or both. This strong assumption will make almost impossible a dividend derivatives smile calibration. The scope of this paper is to present two models that are flexible enough to provide an overarching calibration to dividend futures and dividend European options. One model focuses on the dividend individual payments as jumps associated with the evolution of the underlying stock price. Both the size and frequency is captured from historical data and then the incomplete market model is completed with the futures prices. The second model aims to capture the dynamics of the cum-dividend value within the year. This is very useful since the dividend derivatives payoffs are functions of the cum-dividend itself. For both models we show how to calibrate the smile across the maturities going up to ten years ahead.

The article is structured as follows. Section 2 provides a literature review of dividend modeling related literature. Section 3 describes the data used in this research. The main modeling results are contained in Section 5. Numerical results using the available data are provided in Section 6 and the last Section concludes.

2 The Linkage Between Equity and Dividends

Black (1990) argued that investors value equity by predicting and discounting dividends. Using tests based on volatility, Shiller (1981, 1986) emphasized that stock price movements cannot be explained only from the information on the future dynamics of dividends around a long-run historical trend. In the finance literature the overwhelming conclusion is that future dividends are uncertain, both in their timing and size.

For dividend derivatives it would be useful to know the dividends that will be paid to a given horizon. Harvey and Whaley (1992) and Brooks
(1994) extracted implied dividends employing the put-call parity but these estimators were too noisy for predicting the next dividend. Implied dividends have been utilised as part of the estimation process for risk-neutral densities by Ait-Sahalia and Lo (1998). The empirical properties of dividends have been amply discussed by van Binsbergen et al. (2012).

Practitioners\textsuperscript{1} used to deal with dividend paying stocks by assuming known dividends, in cash or as an yield, and then proceed with an option pricing calculator such as Black-Scholes for example, with a deflated stock price resulting from stripping out the presumed known dividends over the life of the option from current stock price.

Dividends impact on the valuation of financial assets such as plain stock and options. Models that forecast dividends have had mixed results in the literature and the empirical evidence is divided on their usefulness, particularly for long maturities. Chance et al. (2000) developed a forecasting model for dividends taking into account seasonality and mean reversion effects, showing that it is possible to produce unbiased estimators of dividend \textit{related} quantities. Other recent papers proposing various approaches to forecast the dividend yields are van Binsbergen et al. (2012), Chen et al. (2012), Kruchen and Vanini (2008), Buehler et al. (2010).

Chance et al. (2000) analysed index option prices based on ex-post realized dividend information, with the corresponding options valued using ex-ante dividend forecasts, and they found that the latter does not lead to biased pricing, although the sample error is quite large. On the other hand, the implied dividends from S&P500 options may improve significantly the forecasts of market returns as demonstrated by Golez (2014). Using data between 1994 and December 2009, Golez first shows that the dividend-price ratio gives a poor forecast for future returns and dividend growth. Then, a model-free formula for the implied dividend yield is determined from index futures cost-of-carry formula and the put-call parity. The implied dividend yield is then combined with the realized dividend-price ratio to calculate the implied dividend growth and an adjusted dividend-price ratio that have substantial predictive power, in-the-sample and out-of-sample, for market returns.

In their seminal paper on the call option pricing model Black and Sc-\textsuperscript{1}Some interesting readings in this area can be found in Bos and Vandermark (2002), Bos et al. (2003), Frishling (2002), de Boissezon (2011), Lu and Karaban (2009), Manley and Mueller-Glissmann (2008)
holes did not take into account dividend payments on the underlying stock and did not allow for the possibility of early exercise which may be optimal when the stock pays dividends. However, Black (1975) suggested that the original Black and Scholes model can be modified to take account of dividends. Roll (1977) solved the problem of valuing an American call written on an underlying stock with known dividends, observing that the call can be valued as a portfolio of three European options and hence providing an exact formulation. The Roll model does not require the assumption of a continuous dividend generating process as assumed by Merton (1990) Sterk (1982) compared Roll’s model with a dividend adjusted Black-Scholes model calculating the option value from the Black-Scholes formula after reducing the stock price by the present value of the actual dividend payment on the underlying stock, then recalculating the option value assuming that the option is exercised just before the ex-dividend date and take the larger of the two calculated option values. The latter method was only an approximation which did not work well in practice. The first attempt to take into account correctly the impact of uncertain dividend yield on equity option pricing was due to Geske (1978) who provided an adjusted Black-Scholes formula. Moreover, Geske pointed out that assuming that dividends are known when in fact they are not, has the effect to mis-estimate the volatility.

Nevertheless, Chance et al. (2002) demonstrated that when dividends are stochastic and discrete such that the present value of all future dividends is observable and tradable in a forward contract, Black-Scholes formula still applies for pricing European options. The assumption that the present value of all dividend payments generated by a stock to a given maturity is known may seem a quite strong, although the very nascency of dividend futures markets may provide a good mechanism to ascertain this value. The inter-link between dividends and volatility has been emphasized by Broadie et al. (2000), who proved that both dividend risk and volatility risk are relevant for pricing American options contingent on an asset that has both a stochastic volatility and an uncertain dividend yield. Schroder (1999) described a change of numeraire method for pricing derivatives on an underlying that generates dividends. Expanding on this idea Nielsen (2007) proposed a robust theoretical framework for treating dividends when pricing derivatives.

Importantly, Lioui (2006) showed that stochastic dividend yields may lead to a different type of put-call parity, from the one that is normally used to reverse engineer the dividend yield from market European option prices. Recognizing that in practice dividends on stocks are not paid continuously but at discrete times, Korn and Rogers (2005) developed a general approach for stock option pricing, where the absolute size of the dividend is random but its relative size is still constant. One advantage of their model can be adapted to deal with dividends announced in advance and with changing in dividend policy. Bernhart and Mai (2012) generalized this line of modeling dividends as a discrete cash-flow series and proposed a no-arbitrage methodology capable of embedding many well-known stochastic processes and general dividend specification. Brockhaus (2014) presented a more general family of models treating dividends for equity derivatives, encompassing the model in Korn and Rogers (2005).

Another interesting approach in Buehler et al. (2010) considers an equity stock price model with discrete stochastic proportional dividends. Their model assumes that dividend ratios are a linear combination between the classic known proportional dividends and a stochastic dividend part described by an Ornstein-Uhlenbeck process.

3 Data Description

The Dow Jones Euro STOXX50® index dividend futures contract traded on Eurex has a value of 100 EUR per one index dividend point. The contract is cash settled on the first exchange day after the settlement day which is the third Friday of December of each maturity year\(^2\). The minimum price change is 0.1 points and since May 4, 2009 there are ten annual contracts. The final settlement price in this futures contract is determined by the final value of the underlying Dow Jones EURO STOXX50 DVP®, the index of dividends calculated for that annual period. Only gross unadjusted dividends that are declared and paid in the contract period by any of the individual components of the Dow Jones Euro STOXX50® are considered for settlement purposes\(^3\).

\(^2\)If the third Friday is not an exchange day then the settlement day is the exchange day immediately preceding that day.

\(^3\)The settling at maturity is done versus the weighted sum of the gross cumulative cash dividends paid by each company that is part of the Dow Jones Euro STOXX50® index during that period, multiplied by the number of free-float adjusted shares, and the total
The gross ordinary dividends are the unadjusted cash dividends paid between the third Friday of December in preceding year, excluding, and the third Friday of December of current year, inclusive. The futures prices are quoted daily. Hence, index companies paying multiple dividends will contribute on each ex-dividend date based on the free float adjusted share.

The descriptive statistics for the Dow Jones Euro STOXX50® and its corresponding cum-dividend series in index points are presented in Table 1. The standard deviation of the equity index is equal to 8.67 and this is more than five times larger than the standard deviation of the cum-dividend series, in line with the conclusions from Shiller (1981). The time series of paid dividends for Dow Jones Euro STOXX50® is presented in Figure 3. Dividends are measured in index points. One clear characteristics of this data is that it looks like a jump process and that the size of the jumps look stochastic.

Table 1: Descriptive Statistics for the Dow Jones Euro STOXX 50 index and its corresponding cum-dividend series in index points. The historical series covers the daily data for the period 22 December 2008 and 17 December 2012. Source: Eurex.

<table>
<thead>
<tr>
<th></th>
<th>STOXX50 index</th>
<th>CumDividend</th>
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<tbody>
<tr>
<td>Mean</td>
<td>2579.89</td>
<td>64.95</td>
</tr>
<tr>
<td>Standard Error</td>
<td>8.67</td>
<td>1.39</td>
</tr>
<tr>
<td>Median</td>
<td>2592.71</td>
<td>88.27</td>
</tr>
<tr>
<td>Mode</td>
<td>2487.08</td>
<td>7.60</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>278.08</td>
<td>44.43</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>-0.95</td>
<td>-1.63</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.29</td>
<td>-0.38</td>
</tr>
<tr>
<td>Minimum</td>
<td>1809.98</td>
<td>0.00</td>
</tr>
<tr>
<td>Maximum</td>
<td>3068.00</td>
<td>124.34</td>
</tr>
<tr>
<td>Count</td>
<td>1028</td>
<td>1028</td>
</tr>
</tbody>
</table>

is then divided by the index divisor.
Figure 3: The daily dividends in index points paid on the Dow Jones Euro STOXX50® index. The series is daily between 22 December 2008 and 17 December 2012. Source: Eurex.

Another way dividends are reported is based on the cum-dividend series within each calendar market year. This is helpful for the dividend futures contracts traded on Eurex or for index dividend swaps contracts traded over-the-counter. The cum-dividend series depicted in Figure 4 display a very interesting regular pattern. The shape is clearly sigmoidal with an inflection point almost half-way in June. Our stochastic logistic diffusion model described in Section 5.3 is capable of producing exactly this type of dynamics pattern.
Figure 4: The cum-dividend daily time series in index points paid on Dow Jones Euro STOXX50® index. The historical data is daily for the period 22 December 2008 and 17 December 2012. Source: Eurex

The descriptive statistics of the dividend futures prices are reported in Table 2. The last three maturities of the currently ten contracts traded actively on Eurex from 4 May 2009, and in general these three contracts have very similar prices. The graph in Figure 5 shows the futures settlement prices on Dow Jones Euro STOXX50 DVP® index from Eurex for the first seven maturities, using a longer historical data. The nearest maturity contract has had a different evolution compared to the remaining six maturities futures depicted⁴, which have a more correlated dynamics. The only time when they all seem to converge is at rollover time when the pull to maturity effect is noticeable.

For pricing purposes discount factors to the required maturity are also needed. In the aftermath of the subprime crisis the role of the funding rate

⁴To an extent the second maturity dividend futures contract also departs from the rest.
Table 2: Descriptive Statistics for the Futures on Dow Jones Euro STOXX50 DVP® index for all maturities. The historical series is daily between 22 December 2008 and 8 February 2012 for the first seven maturities and between 1 May 2009 and 8 February 2012 for the last three yearly maturities. Source: Eurex.

<table>
<thead>
<tr>
<th></th>
<th>F1</th>
<th>F2</th>
<th>F3</th>
<th>F4</th>
<th>F5</th>
<th>F6</th>
<th>F7</th>
<th>F8</th>
<th>F9</th>
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<tr>
<td>Mean</td>
<td>115.56</td>
<td>102.05</td>
<td>96.42</td>
<td>94.55</td>
<td>94.38</td>
<td>94.96</td>
<td>95.79</td>
<td>100.44</td>
<td>101.37</td>
<td>102.20</td>
</tr>
<tr>
<td>s.e.</td>
<td>0.22</td>
<td>0.65</td>
<td>0.65</td>
<td>0.62</td>
<td>0.61</td>
<td>0.60</td>
<td>0.60</td>
<td>0.54</td>
<td>0.56</td>
<td>0.58</td>
</tr>
<tr>
<td>Median</td>
<td>113.45</td>
<td>107.70</td>
<td>99.50</td>
<td>98.30</td>
<td>99.00</td>
<td>100.10</td>
<td>103.60</td>
<td>104.80</td>
<td>106.30</td>
<td></td>
</tr>
<tr>
<td>Mode</td>
<td>112.80</td>
<td>113.90</td>
<td>88.60</td>
<td>104.40</td>
<td>114.80</td>
<td>115.20</td>
<td>105.60</td>
<td>105.50</td>
<td>78.60</td>
<td>111.60</td>
</tr>
<tr>
<td>Std</td>
<td>6.26</td>
<td>18.50</td>
<td>18.42</td>
<td>17.55</td>
<td>17.35</td>
<td>17.03</td>
<td>17.14</td>
<td>14.45</td>
<td>15.04</td>
<td>15.42</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>-0.19</td>
<td>0.49</td>
<td>-0.15</td>
<td>-0.53</td>
<td>-0.72</td>
<td>-0.85</td>
<td>-0.89</td>
<td>-1.02</td>
<td>-0.97</td>
<td>-0.95</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.02</td>
<td>-1.18</td>
<td>-0.84</td>
<td>-0.64</td>
<td>-0.52</td>
<td>-0.44</td>
<td>-0.42</td>
<td>-0.33</td>
<td>-0.35</td>
<td>-0.39</td>
</tr>
<tr>
<td>Min</td>
<td>96.10</td>
<td>54.00</td>
<td>51.70</td>
<td>53.70</td>
<td>54.50</td>
<td>55.50</td>
<td>57.20</td>
<td>69.90</td>
<td>69.50</td>
<td>69.20</td>
</tr>
<tr>
<td>Max</td>
<td>125.10</td>
<td>125.30</td>
<td>119.90</td>
<td>120.60</td>
<td>122.90</td>
<td>124.30</td>
<td>126.50</td>
<td>128.80</td>
<td>131.10</td>
<td>132.50</td>
</tr>
<tr>
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<td>806</td>
<td>806</td>
<td>806</td>
<td>806</td>
<td>806</td>
<td>717</td>
<td>717</td>
<td>717</td>
</tr>
</tbody>
</table>

has become prominent. Since the purpose of our paper is to price dividend derivatives traded on an exchange there is no reason for a collateral agreement and therefore for using OIS curve for discounting. Hence, here we work with discount factors calibrated from the Euribor-swaps curve. In order to have a smooth pasting from euro futures implied rates to swap implied rates we use 3-month tenor swaps with a 3-month Euribor reference rate.

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Figure 5: The futures curves for Dow Jones Euro STOXX50 DVP® index. The daily series for the first seven yearly December maturities are presented for the period 22 December 2008 to 8 December 2012. *Source:* Eurex.

4 Previous Modeling of Dividends

In general pricing dividend derivatives has been done so far in two ways: a model-free financial engineering approach described next, and a bottom-up econometric driven approach whereby analysts use data driven methods to forecast the future dividends and their time. The former is applied more when the underlying is the dividend stream of an equity index whereas the latter is favored by investors trying to find statistical arbitrage opportunities.

Denoting with $\text{Div}_{t,T}$ the gross dividend paid on the equity index over the period $[t, T]$ the *forward* price at time $t$ on the cumulative dividend stream $\{\text{Div}_{t,T}\}_{t \leq u \leq T}$ is given by $FW_t(\text{Div}_{t,T}) = PV_t(\text{Div}_{t,T}) \exp (r_{t,T} (T - t))$ where $r_{t,T}$ is the risk-free interest rate and $PV_t(\text{Div}_{t,T})$ is the present value of the
gross dividend stream for the period \([t, T]\) at time \(t\). The put-call parity for European options was employed as a model-free way to calculate the implied dividend quantity to the required horizon of dividend derivatives. Thus

\[
P V_{t}(D iv_{t,T}) = S_{t} + p_{t}^{E}(K, T) - c_{t}^{E}(K, T) - K \exp \left[-r_{t,T}(T - t)\right].
\]

(1)

where \(S\) is the underlying equity index, \(c_{t}^{E}\) and \(p_{t}^{E}\) are the call and put European option prices with maturity \(T\) and exercise price \(K\). This approach is assuming the risk-free rate as given. Considering \(q_{t,T}\) as the continuously compounded dividend yield for the period \([t, T]\) Golez (2014) suggested reverse engineering both the implied risk free rate \(r_{t,T}\) and the implied dividend yield \(q_{t,T}\) from the futures price formula

\[
F_{t}(T) = S_{t} \exp \left[\left(r_{t,T} - q_{t,T}\right)(T - t)\right] - K \exp \left[-r_{t,T}(T - t)\right] - c_{t}^{E}(K, T) + p_{t}^{E}(K, T)
\]

(2)

and

\[
F_{t}(T) = S_{t} \exp \left[\left(r_{t,T} - q_{t,T}\right)(T - t)\right] - K \exp \left[-r_{t,T}(T - t)\right] - c_{t}^{E}(K, T) + p_{t}^{E}(K, T)
\]

(3)

The two equations (2) and (3) give

\[
r_{t,T} = \frac{1}{T - t} \log \left[\frac{F_{t}(T) - K}{c_{t}^{E}(K, T) - p_{t}^{E}(K, T)}\right]
\]

(4)

and

\[
q_{t,T} = \frac{1}{T - t} \log \left[\frac{\left(c_{t}^{E}(K, T) - p_{t}^{E}(K, T)\right)}{S_{t}} + K \left(\frac{c_{t}^{E}(K, T) - p_{t}^{E}(K, T)}{F_{t}(T) - K}\right)\right]
\]

(5)

Because \(PV_{t}(D iv_{t,T}) = \exp \left[-q_{t,T}(T - t)\right]\) the value of the forward price on the dividends on Dow Jones Euro STOXX50® follows immediately. Alternatively one can reverse-engineer from put-call parity directly the present value of all gross returns

\[
PV(D iv_{t,T}) = S_{t} - \left[c_{t}^{E}(K, T) - p_{t}^{E}(K, T)\right] \frac{F_{t}}{F_{t} - K}.
\]

(6)

This model free approach, while easy to implement, has several flaws. First and foremost, it assumes that the future dividends are known to some

\[5\text{Remark that in this case, due to the tacit assumption that } r_{t,T}\text{ is constant, forward prices are equal to futures prices.}\]
extent, which is not the case in reality. Secondly, the dividend derivatives
payoff takes into consideration only the gross unadjusted dividends that are
declared and paid in the contract period. There could be other dividend pay-
ments involved with a given company that may impact on the current price
of European options on the shares of that company. Therefore, the options
prices may encapsulate dividend information that is not corresponding one-
to-one with the gross unadjusted dividends. Another problem is the choice
of the exercise price $K$ since the results may be sensitive between the futures
price series and interest rate series to see if the futures prices are equal to
forward prices.

Frishling (2002) discussed three different approaches to model the link-
age between dividends and stocks modeled stochastically. Consider that
the future dividend dates are given generically at discrete time $t_i$, where
$t < t_1 < \ldots < t_N < T$ and $t$ is today and $T$ denotes future maturity. The
first approach is the escrowed model given by

$$
\begin{align}
    dC_t &= rC_t dt + \sigma C_t dW_t \\
    S_t &= C_t + \sum_{t < t_i < T} D_{t_i} e^{-r(t_i-t)} \\
    S_T &= C_T
\end{align}
$$

(7)

where $\{S_t\}_{0 \leq t \leq T}$ is the stock price process, $\{C_t\}_{0 \leq t \leq T}$ is the capital price
process and $D_{t_i}$ is the fixed lump sum dividend paid at time $t_i < T$, and of
course $r$ is the constant risk-free rate. Although the process $\{S_t\}_{0 \leq t \leq T}$ is not
a geometric Brownian motion, the process $\{C_t\}_{0 \leq t \leq T}$ is, and then the Black-
Scholes model can be applied to the latter with $C_0 = S_0 - \sum_{t < t_i < T} D_{t_i} e^{-r(t_i-t)}$.

The second model has been described more formally in Musiela and
Rutkowski (1997) and it is linked to an idea of working with an accumu-
lation process rather than dividend stripped process. The model is described
by the equations

$$
\begin{align}
    dA_t &= rA_t dt + \sigma A_t dW_t \\
    S_t &= A_t - \sum_{0 < t_i < t} D_{t_i} e^{r(t_i-t)} \\
    S_0 &= C_0
\end{align}
$$

(8)

Once again the stock price process $\{S_t\}_{0 \leq t \leq T}$ is not a geometric Brownian
motion but the accumulator process $\{A_t\}_{0 \leq t \leq T}$ is and for contingent claims
on stock one can work with the latter. Dividends are again assumed to be
known in size and timing.
The third model is a standard jump-diffusion model with *deterministic* jumps arriving at pre-specified times

\[
\begin{dcases}
\quad dS_t = rS_t dt + \sigma S_t dW_t \\
\quad S_{t_i} = S_{t_{i-}} - D_{t_{i}},
\end{dcases}
\tag{9}
\]

This model is not lognormal because of the discontinuity at dividend paying time \( t_i \).

Frishling (2002) showed via an example that for the same dividend payment and identical parameters for stock price it is possible to get very different distributions for the stock at maturity \( T \) when using different models. Hence, the method employed to model dividends can have a great impact on the final results.

5 Models for Index Dividend Derivatives

Previous studies, see Baldwin (2008), have hinted that dividend yields implied by the Dow Jones Euro STOXX50® Index dividend swap contracts are uncorrelated to the three-month EURIBOR rates. Here we have tested this analysis for the period 23 December 2008 to 8 February 2012 and for the first six maturities of the Eurex dividend futures contracts. The OLS regression lines depicted on each graph all have very low \( R^2 \) values, confirming previous conclusions that interest rates are uncorrelated to dividend futures prices.

This empirical artefact supports the idea that futures prices may be congruent with forward prices in the case of Dow Jones Euro STOXX50® dividend index. Thus the futures prices are given by the expectation of the payoff under a suitable risk-neutral measure.

Remark that it is possible to have a low \( R^2 \) value but the explanatory regression variable to be significant. Hence, for each December maturity the null hypothesis that the changes in Euribor rates do not impact upon the changes on implied dividend yields was tested. In all cases, we have failed to reject the null hypothesis.
Figure 6: Scatter plots of daily changes in dividend futures implied yields and the corresponding three month EURIBOR funding rates. Data for the period 23 December 2008 to 8 February 2012. The daily first differences in implied dividend yields are on the vertical axis in index points while the daily changes in 3-month Euribor funding equivalent rate to the maturity of the corresponding futures contract are on the horizontal axis.

For pricing and calibrating dividend index derivatives we consider a time
grid given by
\[ T_0 < t_0 < t_1 \ldots < t_{n_1} < \ldots < T_1 < \ldots < T_2 < \ldots < T_{10} < \ldots < T^* \]
where \( T^* \) is a very large but still finite maturity, \( T_i \) are yearly December maturities with \( i = 1, \ldots, 10 \), and \( t_j \) are daily times so \( t_{j+1} - t_j = \Delta t \) represents one day, for any positive integer \( j \) and \( T_{i+1} - T_i = 1 \) year, for any \( i \).

5.1 A jump-diffusion model for dividends

The first model analysed here is a jump-diffusion model with jumps tailored for dividends only. Thus, the jumps can be only downward jumps. The dividend payments are intrinsically linked to the corresponding equity index. The dynamics therefore should follow the equity index. Under the physical measure \( \mathbb{P} \)

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} [V_i - 1] \right) \tag{10}
\]

where \( \{W_t\}_{0 \leq t \leq T^*} \) is a Wiener process, \( \{N_t\}_{0 \leq t \leq T^*} \) is a Poisson process with arrival rate \( \theta \) accounting for the payment times of dividends per unit of time and \( \{V_i\}_{i \geq 1} \) are i.i.d with distribution function \( H \) representing the jump sizes. The three stochastic structures are assumed to be mutually independent. As it is standard, \( \mu \) is a real constant and \( \sigma \) is a positive number.

The SDE (10) has the solution

\[
S_t = S_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \prod_{i=1}^{N_t} V_i \tag{11}
\]

or, slightly more generally

\[
S_T = S_t \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma \sqrt{T - t} Z \right\} \prod_{i=N_t}^{N_T} V_i \tag{12}
\]

with \( Z \sim N(0, 1) \). For the approach proposed here the following assumptions are made.

Assumption 5.1. All jumps in the equity index dynamics are downward, reflecting dividend adjustments.
**Assumption 5.2.** All dividends are in index points and are a stochastic proportion of the equity index.

Hence, this model lies between the usual jump-diffusion models for equity asset pricing due to Merton (1990) and the jump to default credit risk models. The jump sizes here are random quantities $V_i \in (0, 1)$. The price of the index ex-dividend is $S_tV_t$ so the dividend paid for day $t$ is $S_t(1 - V_t)$, and this will be paid with probability $\theta \Delta t$. In order to simplify the notation, $\delta_t \equiv 1 - V_t$ henceforth. Thus, the cum-dividend in index points for the period $(t, T]$ is given by

$$
Div_{(t,T]} = \sum_{j=1}^{j=m} S_{t+j\Delta t} \delta_{t+j\Delta t} + \sum_{j=1}^{k} Y_{t+j\Delta t} + \sum_{j=1}^{k} (13)
$$

where $m = \frac{T-t}{\Delta t}$ and $t \equiv t_k$, and $\{Y_{t+j}\}_{j\geq1}$ are Bernoulli variables taking the value 1 with probability $\theta \Delta t$ and the value zero with probability $1 - \theta \Delta t$. The pricing of any contingent claim on $Div_{(t,T]}$ can be carried out by Monte Carlo simulation under a suitable risk-pricing measure.

The model presented so far is quite general and it covers a wide range of specifications that depend further on how jumps are viewed in relation to the underlying index and also on various distributions for the jump sizes such that jumps are only downward. Here we propose a full parametric approach and therefore make a third assumption about the distribution of the jump sizes.

**Assumption 5.3.** $\{V_i\}_{i\geq1}$ are i.i.d and $V_i \sim Beta(a, b)$.

If $V \sim Beta(a, b)$ then $\delta = 1 - V$ is distributed with $Beta(b, a)$. Then, for any $k$ and any $j$

$$
\mathbb{E}_{t_k}(\delta_{t+j\Delta t}) = \frac{b}{a+b}
$$

Under our modeling assumptions, as in Merton (1990) and Duffie (1995), a unique risk-neutral measure $\mathbb{Q}$ is the one associated with the SDE

$$
\frac{dS_t}{S_t} = (r - \theta E(V - 1))dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}[V_i - 1]\right) (14)
$$

While here it is assumed that jumps are fully diversifiable and therefore jump risk is non-systematic, other models may specify a relationship between jumps and risk-preferences of the market representative investors.
where \( r \) is the risk-free rate assumed constant. Given our parametric assumption of a beta distribution for the jump sizes, the SDE under the risk-neutral pricing measure is

\[
\frac{dS_t}{S_t} = \left( r - \theta \frac{b}{a + b} \right) dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} [V_i - 1] \right). \tag{15}
\]

While this equation cannot be used for the dividends themselves, it is still necessary for this model because the future dividend payments are proportional payments \( \delta_\tau \) of the corresponding equity index \( S_\tau \) at each dividend payment time \( \tau \).

5.2 Calibration Methodology of the Jump-Diffusion Model for Dividends

For practical purposes we need to estimate the parameters \( a, b, \sigma, \theta \) driving the dynamics of the downward jump-diffusion model with beta distributed jumps in equation (15). The arrival rate \( \theta \) and the parameters \( a, b \) calibrating the jump-size can be estimated from the daily dividend payment series.

For the parameter \( \sigma \) one can estimate it directly from the time series of equity index after filtering out the days when dividends were paid. The total volatility would then have two components, one given by the index and one by dividend jumps. Alternatively, the implied volatility gauged from the options traded on the equity index can be used, although that approach has not been used here.

The risk-free rate is considered here as a constant\(^8\) approximating the cost of funding to the required horizon. Different values are used for different horizons and the risk-free rate is calibrated from Euribor-swap market curve on the day of calculation.

For pricing futures and European options a Monte Carlo approach is followed that simulates daily paths to the required maturity. Each day we simulate possible values from a standard geometric Brownian motion under the risk-neutral pricing measure. This is equivalent to using the continuous time diffusion part in (15). Then, we simulate in a binary fashion whether a

\(^8\)A more elaborated approach would involve having a separate short-rate model or market model for the risk-free rate. Given that post subprime-liquidity crisis it is difficult to say which model would be most appropriate for interest free rate concept, we prefer to use a unique number for \( r \).
dividend payment is made. The probability of success is equal to \( \theta \Delta t \). Conditional on a dividend payment being made a random draw from a \( \text{Beta}(b, a) \) distribution is made for the size of the jump. If a dividend payment is made the value of the simulated equity index is reduced proportionately exactly with the size of the jump.

This methodology has the advantage that once paths are simulated to required maturities, any other derivatives, including path dependent derivatives, can be priced accordingly. A similar procedure can be implemented to produce risk measures, such as value-at-risk, derived from the dynamics of the model presented in this section, under the physical measure.

5.3 A Stochastic Logistic Diffusion Model

From the graph in Figure 4 the cumulative dividends time series paid on the Dow Jones Euro STOXX50\(^\circledast\) index displays an interesting stationarity and yearly periodicity. The most striking characteristic is the sigmoidal shape of the series within each year and the fact that there is an acceleration of dividend payments followed by a change of convexity during the period May-June.

It would seem useful if one could model directly the cum-dividends series. In this section we denote by \( X_t \) the cum-dividend from the beginning of the year \( T_{i-1} \) until the current time \( t \), where \( t \leq T_i \), and \( i = \{1, \ldots, 10\} \), with \( T_0 = 0 \).

Under the physical measure \( \mathbb{P} \) the main model proposed in this research is characterised by the following SDE

\[
dX_t = \nu X_t \left( 1 - \frac{X_t}{F} \right) \, dt + \sigma X_t \, dW_t^\mathbb{P}.
\]

(16)

This is the stochastic diffusion version of the Verhulst-Pearl differential model describing constrained growth in biology \(^9\). This model has been called also the geometric mean reversion model. It appeared in finance literature early on but financial research on it has been sparse so far. Merton (1975) arrived at this process looking at the output-to-capital ratio derived from a growth

---

\(^9\)The model was called the logistic growth model because it gives the dynamics of a population which grows at a geometric rate in an environment with limited feeding resources. In that context \( b \) denotes the growth rate per individual, \( F \) is the maximal level of population that can be supported by the resources in the environment and \( \sigma \) is a variation parameter.

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model with uncertainty based on a Cobb-Douglas production function and assuming that gross savings are a deterministic fraction of output. The general model discussed by Metcalf and Hasset (1995) contains the model given in (16) as a particular case.

It can be proved, see Appendix, that the solution to the equation (16) is given by

\[ X_t = X_0 \exp \left( (\nu - \frac{\sigma^2}{2}) t + \sigma W_t \right) \left( 1 + \frac{\nu X_0}{F} \int_0^t \exp \left( (\nu - \frac{\sigma^2}{2}) s + \sigma W_s \right) ds \right) \]  

where \( X_0 \equiv X_{T_{i-1}} \) is the initial point. The solution shows that \( X_t > 0 \) at any time \( t \in (T_{i-1}, T_i) \) for any parameters \( \nu, F, \sigma \) and initial starting point \( X_0 \). The interpretation of the parameters is interesting in itself in a dividend market space. The upper limit for the corresponding logistic process\(^{10}\) is \( F \) while \( \nu \) is the speed of production of dividends. As pointed out by Merton (1975) and reinforced recently by Yang and Ewald (2010), for the parameter \( F \) of the stochastic logistic diffusion model it is not true that \( \lim_{t \to \infty} \mathbb{E}^\mathbb{P}(X_t) = F \).

The model given above is in isolation of any dynamics of the equity index itself. This would solve the equity-dividend puzzle discovered by Shiller (1981) that makes equity dynamics incompatible with the production of future dividends from a volatility perspective. Thus, this model should be more robust for pricing dividend derivatives than the previous jump-diffusion model. The stochastic logistic diffusion model described by (16) implies an incomplete market for dividend payments because the underlying is not a tradable asset. Fortunately the dividend futures contracts traded on Eurex are available to complete the market and determine the martingale pricing measure that can be used for pricing other derivatives such as European call and put options. This can be done period by period. Following Bjork (2009) we can fix the martingale measure \( \mathbb{Q} \) by determining the market price of risk \( \lambda(t, X_t) \) such that

\[ dX_t = X_t \left( \nu - \lambda(t, X_t) \sigma - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^\mathbb{Q}. \]  

\(^{10}\)The logistic process is defined purely by the drift so the equation is the following ODE \( \frac{dX_t}{dt} = \nu X_t \left( 1 - \frac{X_t}{F} \right) \) which can be solved analytically to give the logistic function with the well-known sigmoidal shape.
Since at each moment in time \( t \) the market will be completed for all 10 years spanned by the running futures contracts, we assume that \( \lambda(t, X_t) \equiv \lambda_i \), for all \( i = \{1, \ldots, 10\} \). Each parameter \( \lambda_i \) will be identified by exact calibration to dividend futures prices from the model with the dynamics given by the SDE for any \( T_{i-1} < t \leq T_i \)

\[
dX_t = X_t \left( \nu - \lambda_i \sigma - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^Q.
\]

The distribution of the solution in (17) has been derived in closed-form by Yang and Ewald (2010) but it is cumbersome for practical calculations even of vanilla derivatives such as European put and call options.

Nevertheless, the calibration of parameters \( \lambda_i \) can be easily done from futures market prices because the futures price of the payment \( Div_{T_i-1, T_i} \) is equal to \( E^Q(X_{T_i}) \). Thus, the parameter \( \lambda_i \) can be determined by first discretizing the equation (20) into

\[
X_{T_{i-1} + j \Delta t} = X_{T_{i-1} + (j-1) \Delta t} \left[ 1 + \left( \nu - \lambda_i \sigma - \frac{X_{T_{i-1} + (j-1) \Delta t}}{F} \right) \Delta t + \sigma \sqrt{\Delta t} Z_j \right]
\]

where \( Z_j \sim N(0, 1) \), \( \forall j \), and then generating \( M \) different paths between \( X_{T_{i-1}} \) and \( X_{T_i} \), and finally computing the required expectation by Monte Carlo

\[
E^Q_t(X_{T_i}) = \frac{1}{M} \sum_{k=1}^{M} X_{T_i}^{(k)}.
\]

There are two major advantages of this Monte Carlo approach: a) the futures curve provided by the dividend futures market on Eurex will be perfectly calibrated, and b) other derivatives, including path-dependent derivatives, can be directly priced since path values are readily available under the correct martingale measure. From a computational point of view, exact simulation—in the sense that there is no discretization error—of diffusion sample paths can be produced using the algorithm presented in Beskos et al. (2006).

While for the maturities 2 to 10 the simulation exercise is more straightforward since the entire year is used for path simulations of cum-dividends, for the current year care must be taken since at any time \( t > T_0 \) some dividends may have been paid already. This is more relevant for calibration purposes.

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5.4 Calibration Methodology

For calibration purposes we need to calibrate the parameters $\nu, F$ and $\sigma$ from historical time-series, under the physical measure. The model in (16) can be discretized in the following form

$$\frac{X_{t+\Delta t} - X_t}{X_t} = \nu \Delta t - \frac{\nu \Delta t}{F} X_t + \sigma \sqrt{\Delta t} Z$$

(21)

Denoting $R_t = \frac{X_{t+\Delta t} - X_t}{X_t}$ for the return series, under the assumption that $R_t \equiv 0$ when $X_t = 0$, the corresponding regression model can be fit to cum-dividend data within a year

$$R_t = \alpha + \beta X_t + \varepsilon_t,$$

(22)

with $\varepsilon_t \sim N(0, s^2)$. The parameters of the regression model can be linked to the financial engineering model parameters through the following formulae

$$\hat{\nu} = \frac{\hat{\alpha}}{\Delta t}, \quad \hat{F} = -\frac{\hat{\alpha}}{\hat{\beta}}, \quad \hat{\sigma} = \frac{\hat{s}}{\sqrt{\Delta t}}$$

(23)

Remark that if $F$ is considered known then there are only two parameters to calibrate $\nu$ and $\sigma$ and this can be done from the regression through the origin model

$$R_t = \beta Y_t + \varepsilon_t$$

where $Y_t = 1 - \frac{1}{F} X_t$.

6 Numerical Examples

In this section we shall explore some numerical exemplification of the two dividend models proposed in this paper.

6.1 Jump-down diffusion model

The arrival rate of dividends can be estimated very easily from data since the sample mean is the maximum likelihood estimator which is unbiased and also a sufficient statistic. Hence, I have estimated the parameter $\theta$ over three different periods to see if there are large differences. The results presented in Table 3 suggest that the arrival rate estimated from the entire historical
data until the given date is roughly the same although one can argue in favor of a time trend\textsuperscript{11}. The parameter $\sigma$ is calibrated from historical data and we got $\sigma = 21\%$ for the volatility of the Dow Jones Euro STOXX50\textsuperscript{®}.


<table>
<thead>
<tr>
<th>Period</th>
<th>$\hat{a}_{MLE}$</th>
<th>$b_{MLE}$</th>
<th>$\hat{\theta}_{MLE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 Dec 2008 to 18 Dec 2009</td>
<td>8.0254</td>
<td>0.9584</td>
<td>0.1811</td>
</tr>
<tr>
<td>21 Dec 2009 to 17 Dec 2010</td>
<td>9.3934</td>
<td>0.9971</td>
<td>0.1863</td>
</tr>
<tr>
<td>20 Dec 2010 to 17 Dec 2011</td>
<td>10.1200</td>
<td>1.0321</td>
<td>0.1927</td>
</tr>
<tr>
<td>19 Dec 2011 to 17 Dec 2012</td>
<td>10.1790</td>
<td>1.0559</td>
<td>0.1936</td>
</tr>
</tbody>
</table>

Before showing the results of calibration on 20 Dec 2010, we need to introduce a scaling parameter $c$. Preliminary results using Monte Carlo simulation\textsuperscript{12} indicate that applying the model with jump downward dividends has the effect of extreme bias in calibrating the dividend futures prices when the dividend adjustments factors are allowed to be between $(0, 1)$. A closer analysis reveals the source of the problem. The downward jump allows jumps close to zero which are equivalent to dividend payments closer to the actual value of the index. In practice this is not true and the parameter $c$, with $c < 1$ such that $V \sim c \times \text{Beta}(a, b)$, allows rescaling dividends to a more suitable range $(0, c)$ rather than the $(0, 1)$ range of the standard beta distribution. This parameter can be finely tuned to calibrate the dividend futures curve. For 20 Dec 2010, $c = 0.625$ seems to calibrate the options prices well for all maturities. The results for pricing the European call and put prices for the first four maturities are displayed in Figure 7. With the exception of the put prices for the 16 Dec 2011 maturity, the fit is remarkable.

\textsuperscript{11}We have also estimated the arrival rate at two random points in time and we got $\hat{\theta} = 0.1573$ for 30 Apr 2009 and $\hat{\theta} = 0.1900$ for 8 Feb 2012. This values provide some evidence against a time trend.

\textsuperscript{12}These are not shown here due to lack of space.
Figure 7: European Option pricing with the downward jump-diffusion beta dividend model for first four December maturities for the indicated maturities.
The option pricing results on the same day for the remaining six maturities are illustrated in Figure 8. Overall the smile calibration is excellent.

Figure 8: European Option pricing with the downward jump-diffusion beta dividend model for first four December maturities for the indicated maturities.
6.2 Stochastic Logistic Diffusion Model

Following the methodology presented in Section 5.3 the parameters \( \nu, F \) and \( \sigma \) are calibrated from the OLS estimates of the corresponding linear regression models over one year of data. The results presented in Table 4 indicate a good parameter stability, although the variance has been reduced somehow for the last year.

Table 4: Estimation of parameters for daily cum-dividend series by OLS estimation of simple linear regression model with daily data. \( s^2 \) is the residual sum of squares used to estimate the variance of the regression. **Source of data:** Eurex.

<table>
<thead>
<tr>
<th>Period</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( s^2 )</th>
<th>( \nu )</th>
<th>( F )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 Dec 2008 to 18 Dec 2009</td>
<td>0.0553</td>
<td>-0.0005</td>
<td>0.0123</td>
<td>19.9264</td>
<td>110.61</td>
<td>2.10</td>
</tr>
<tr>
<td>21 Dec 2009 to 17 Dec 2010</td>
<td>0.0588</td>
<td>-0.0005</td>
<td>0.0125</td>
<td>21.1892</td>
<td>104.71</td>
<td>2.12</td>
</tr>
<tr>
<td>20 Dec 2010 to 16 Dec 2011</td>
<td>0.0601</td>
<td>-0.0005</td>
<td>0.0168</td>
<td>21.6541</td>
<td>118.71</td>
<td>2.46</td>
</tr>
<tr>
<td>19 Dec 2011 to 17 Dec 2012</td>
<td>0.0404</td>
<td>-0.0003</td>
<td>0.0048</td>
<td>14.5683</td>
<td>115.57</td>
<td>1.32</td>
</tr>
</tbody>
</table>

Table 5: Estimation of parameters for daily cum-dividend series by OLS estimation of simple linear through origin regression model with daily data under the assumption that \( F = 120 \). **Source of data:** Eurex.

<table>
<thead>
<tr>
<th>Period</th>
<th>( \beta )</th>
<th>( s^2 )</th>
<th>( \nu )</th>
<th>( F )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 Dec 2008 to 18 Dec 2009</td>
<td>0.0537</td>
<td>0.0123</td>
<td>19.3406</td>
<td>120</td>
<td>2.10</td>
</tr>
<tr>
<td>21 Dec 2009 to 17 Dec 2010</td>
<td>0.0559</td>
<td>0.0125</td>
<td>20.1248</td>
<td>120</td>
<td>2.12</td>
</tr>
<tr>
<td>20 Dec 2010 to 16 Dec 2011</td>
<td>0.0599</td>
<td>0.0167</td>
<td>21.5976</td>
<td>120</td>
<td>2.45</td>
</tr>
<tr>
<td>19 Dec 2011 to 17 Dec 2012</td>
<td>0.0399</td>
<td>0.0048</td>
<td>14.3713</td>
<td>120</td>
<td>1.32</td>
</tr>
</tbody>
</table>

The estimation results in Tables 4 and 5 indicate that parameters may change but not very much. The greatest variation year on year seems to occur for parameter \( F \). A more robust estimation process using Bayesian inference may improve the accuracy of the parameters estimators for the stochastic logistic diffusion model. The parameters estimated from data over one year will be kept constant for all derivatives calculations during the subsequent year.
Figure 9: Term structure of market price of risk parameter $\lambda$ for all ten December maturities calibrated from Eurex market futures prices on the indicated days. The calibration is done by matching the dividend futures market prices with the theoretical dividend futures given by the stochastic logistic diffusion model.

In Figure 9 the term structure of market price of risk parameter $\lambda$ are illustrated for three different days. These values are calculated at the beginning of the December roll and they are fixing the martingale pricing measure for each of the ten December maturities. The shape of the term structure of market price of risk for Dow Jones Euro STOXX50 DVP® can be inverted,
upward trending and upward then downward trending. Overall the curves presented in Figure 9 suggest that the term structure of $\lambda$ is almost always concave, but this is more a conjecture at this stage of research in this area. Once the pricing measure is determined by calibrating the futures curves, all other contingent claims on the Dow Jones Euro STOXX50® dividend index can be calculated directly. Applying the Monte Carlo methodology described in Section 5.3 it is possible to determine the price of European call and put options, as well as other path dependent derivatives.

The graphs in Figures 10 and 11 depict the smile fit for European options on Dow Jones Euro STOXX50® dividend index on 20 Dec 2010 based on market data from Eurex. First, the estimated parameters from the historical evolution of dividends paid on the Dow Jones Euro STOXX50® index between 21 Dec 2009 and 19 Dec 2010, are used. The smile fit is very good overall, considering the small number of parameters underpinning the stochastic logistic diffusion model. If parameter $F$ is fixed to 120, the smile fit exhibited in Figure 11 indicates almost a perfect fit, only the nearest maturity showing a worsening in smile fit. This may suggest that the representative market agent is using $F = 120$ as indicative for upper limit of dividends in this market!

Once again the nearest maturity seems to be the hardest to calibrate. One explanation advanced by practitioners is that many traders strongly believe in their short-term forecast of dividends and therefore there is very little if any volatility induced by the market view on the nearest maturity puts. Thus, the difference between the market put price curve and the model put price curve represents the volatility that is not taken into account because of the traders beliefs of what the dividends payments will be. Nevertheless, taken into account the simplicity of this model, using only three parameters, the smile calibration across maturity range is excellent.

13Because of space restrictions here we show only the first eight maturities; however the graphs for all ten maturities are available upon request from the author.
Figure 10: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2, F = 104.7, \sigma = 2.12$

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Figure 11: European call and put option price calibrated on 20 Dec 20 10 using $b = 21.2, F = 120, \sigma = 2.12$

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7 Conclusion

The literature on pricing dividend derivatives is sparse. From the equity derivatives pricing literature it seems conclusive that dividends are stochastic in nature. Hence, it is important to find models that can be easily implemented but that also preserve the stochastic character of dividends.

A jump diffusion model with scaled beta distributed jump sizes was proposed for equity dividend index. The jumps are only downwards and the dividend payments are determined also by the evolution of the equity index itself. A Monte Carlo approach was developed for pricing vanilla dividend derivatives. It was illustrated that this model can fit the smile of the European call and put dividend index options.

The stochastic logistic diffusion model is a continuous-time finance model that has not been used very often in finance in the past. The model is very easy to interpret and it calibrates very well the dividend options smile. One great advantage of this model is that it considers directly the dynamics of the dividend index itself, in other words it is suited for dividend derivatives as an asset class of its own.

The two models developed here for pricing dividend derivatives are very different, the first one modeling the dividend payment series while the latter follows the cum-dividend series. Both models rely on the Monte Carlo approach for implementation but there are immediate advantages in doing so since other path dependent derivatives would be priced directly based on the same set of simulations. The stochastic logistic diffusion model for dividend index has the advantage that it is disconnected from the dynamics of the associated stock index. In this way the well-known puzzle identified by Shiller (1981) that no observed movements in the aggregated dividends were ever correctly forecast by movements in aggregate stock prices is circumvented.
References


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A Closed-form solution of stochastic logistic diffusion model

Here we show how to derive the analytical solution of the SDE for the stochastic logistic diffusion model given in the paper by equation (16).

\[
dX_t = bX_t \left(1 - \frac{X_t}{F}\right) dt + \sigma X_t dW_t^P. \tag{24}
\]

Considering the transformation \( Z_t = \frac{F}{X_t} \) we get via Ito’s lemma that

\[
dZ_t = \left[ (\sigma^2 - b)Z_t + b \right] dt - \sigma Z_t dW_t^P \tag{25}
\]

Standard stochastic calculus can be applied to solve directly the linear coefficients SDE of the type

\[
du_t = (a_1 u_t + a_2) dt + b_1 u_t dW_t^P.
\]

The solution is

\[
u_t = \Psi_t \left[ u_0 + a_2 \int_0^t \Psi^{-1}_s ds \right]
\]

where \( \Psi_t = \exp \left( (a_1 - \frac{\sigma^2}{2}) t + b_1 W_t^P \right) \).

Taking \( a_1 = \sigma^2 - b, \ a_2 = b \) and \( b_1 = -\sigma \) implies that

\[
\Psi_t = \exp \left( \frac{\sigma^2}{2} - b \right) t - \sigma W_t^P \]

Hence,

\[
Z_t = \exp \left( \frac{\sigma^2}{2} - b \right) t - \sigma W_t^P \left[ Z_0 + b \int_0^t \exp \left( \frac{\sigma^2}{2} - b \right) s - \sigma W_s^P \right]
\]

which leads to the solution

\[
X_t = \frac{X_0 \exp \left( (b - \frac{\sigma^2}{2}) t + \sigma W_t \right)}{1 + \frac{bX_0}{\sigma} \int_0^t \exp \left( (b - \frac{\sigma^2}{2}) s + \sigma W_s \right) ds}. \tag{26}
\]